Superdeduction

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Understanding proofs

Today's proof assistants approaches:

- tactics, tacticals
- proof terms

In both cases: witnesses do not explain the proof.
We need underlying logics closer to human reasoning.

In "real life" 1

Computation is not even referred as deduction:

$$\sqrt{2+7} = 3$$

$$prime(3) \Rightarrow prime(1+2)$$

$$\ldots$$

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Deduction modulo: put computation inside a congruence

$$\top\text{-R} \frac{}{\vdash 1+2=3} \; (1+2=3) \equiv (1+2=3)$$

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Deduction modulo: put computation inside a congruence

$$\top\text{-R}\ \frac{}{\vdash 1+2=3}\ (1+2=3) \equiv (0+3=3)$$

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$$\ldots$$

Deduction modulo: put computation inside a congruence

$$\top\text{-R}\ \frac{}{\vdash 1+2=3}\ (1+2=3) \equiv (3=3)$$

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Deduction modulo: put computation inside a congruence

$$\top\text{-R}\ \frac{}{\vdash 1+2=3}\ (1+2=3) \equiv (2=2)$$

In "real life" 1

Computation is not even referred as deduction:

$$\sqrt{2+7} = 3$$

$$\mathit{prime}(3) \Rightarrow \mathit{prime}(1+2)$$

$$\dots$$

Deduction modulo: put computation inside a congruence

$$\top\text{-R} \; \frac{}{\vdash 1+2=3} \; (1+2=3) \equiv (1=1)$$

In "real life" 1

Computation is not even referred as deduction:

$$\sqrt{2+7} = 3$$

$$\mathit{prime}(3) \Rightarrow \mathit{prime}(1+2)$$

$$\ldots$$

Deduction modulo: put computation inside a congruence

$$\top\text{-R}\ \frac{}{\vdash 1+2=3}\ (1+2=3) \equiv (0=0)$$

In "real life" 1

Computation is not even referred as deduction:

$$\sqrt{2+7} = 3$$

$$prime(3) \Rightarrow prime(1+2)$$

$$\ldots$$

Deduction modulo: put computation inside a congruence

$$\top\text{-R}\ \frac{}{\vdash 1+2=3}\ (1+2=3) \equiv \top$$

In "real life" 2

Logical arguments are not mentioned:

"Let be x in A.
[...]
Then x is in B.
Hence, A is included in B."

They are hidden inside the structure of theorems/definitions.

A "small" proof

$$
\forall\text{-L}\;\dfrac{
  \forall\text{-L}\;\dfrac{
    \wedge\text{-L}\;\dfrac{
      \Rightarrow\text{-L}\;\dfrac{
        \forall\text{-R}\;\dfrac{
          \Rightarrow\text{-R}\;\dfrac{
            \text{Ax}\;\dfrac{}{\ldots, x \in A \vdash A \subseteq A, x \in A}
          }{\ldots \vdash A \subseteq A, x \in A \Rightarrow x \in A}
        }{\ldots \vdash A \subseteq A, \forall x.(x \in A \Rightarrow x \in A)}
        \qquad
        \text{Ax}\;\dfrac{}{\ldots, A \subseteq A \vdash A \subseteq A}
      }{\ldots, \forall x.(x \in A \Rightarrow x \in A) \Rightarrow A \subseteq A \vdash A \subseteq A}
    }{(A \subseteq A) \Leftrightarrow \forall x.(x \in A \Rightarrow x \in A) \vdash A \subseteq A}
  }{\forall Y.(A \subseteq Y) \Leftrightarrow \forall x.(x \in A \Rightarrow x \in Y) \vdash A \subseteq A}
}{\forall X.\forall Y.(X \subseteq Y) \Leftrightarrow \forall x.(x \in X \Rightarrow x \in Y) \vdash A \subseteq A}
$$

A “small” proof

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\ldots, x \in A \vdash A \subseteq A, x \in A}\;\text{Ax}
    }{\ldots \vdash A \subseteq A, x \in A \Rightarrow x \in A}\;\Rightarrow\text{-R}
  }{\ldots \vdash A \subseteq A, \forall x.(x \in A \Rightarrow x \in A)}\;\forall\text{-R}
  \qquad
  \dfrac{}{\ldots, A \subseteq A \vdash A \subseteq A}\;\text{Ax}
}{
  \dfrac{
    \dfrac{
      \dfrac{\ldots, \forall x.(x \in A \Rightarrow x \in A) \Rightarrow A \subseteq A \vdash A \subseteq A}{(A \subseteq A) \Leftrightarrow \forall x.(x \in A \Rightarrow x \in A) \vdash A \subseteq A}\;\wedge\text{-L}
    }{\forall Y.(A \subseteq Y) \Leftrightarrow \forall x.(x \in A \Rightarrow x \in Y) \vdash A \subseteq A}\;\forall\text{-L}
  }{\forall X.\forall Y.(X \subseteq Y) \Leftrightarrow \forall x.(x \in X \Rightarrow x \in Y) \vdash A \subseteq A}\;\forall\text{-L}
}\;\Rightarrow\text{-L}
$$

Loading and instanciating

Custom inference rules

With the costum rule:

$$\subseteq\text{-R}\ \frac{\Gamma x \in A \vdash x \in B \Delta}{\Gamma \vdash A \subseteq B, \Delta}\ x \notin \mathcal{FV}(\Gamma, \Delta)$$

Custom inference rules

With the costum rule:

$$\subseteq\text{-R}\ \frac{\Gamma x \in A \vdash x \in B \Delta}{\Gamma \vdash A \subseteq B, \Delta}\ x \notin \mathcal{FV}(\Gamma, \Delta)$$

We can build a much shorter (and readable) proof:

$$\subseteq\text{-R}\ \frac{Ax\ \dfrac{}{x \in A \vdash x \in A}}{\vdash A \subseteq A}$$

Consequences

What are the consequences of adding such a rule ?

- Is it sound ?
- Is it complete wrt. the theory ?
- Do we still have a cut elimination procedure and does it strong normalise ?

Superdeduction

- Internalize a theory inside the deduction system
- Inference rules are systematically derived from the axioms
- Good properties of the deduction system are ensured

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Notation:

- Axioms of the form: $\forall \overline{x}.(P \Leftrightarrow \varphi)$ with P atomic.
- We note them $P \rightarrow \varphi$ and call them proposition rewrite rules.

Building the rules

$$\subseteq_{def}: \; A \subseteq B \rightarrow \forall x.(x \in A \Rightarrow x \in B)$$

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$$\subseteq_{def}: \ A \subseteq B \longrightarrow \forall x.(x \in A \Rightarrow x \in B)$$

$$\forall\text{-R}\ \dfrac{\Rightarrow\text{-R}\ \dfrac{x \in A \vdash x \in B}{\vdash x \in A \Rightarrow x \in B}}{\vdash \forall x.(x \in A \Rightarrow x \in B)}\ x \notin \mathcal{FV}$$

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$$\downarrow$$

$$\subseteq_{def}\text{-R}\ \frac{\Gamma, x \in A \vdash_{+} x \in B, \Delta}{\Gamma \vdash_{+} A \subseteq B, \Delta}\ x \notin \mathcal{FV}(\Delta, \Gamma)$$

Building the rules

$$\subseteq_{def}: \; A \subseteq B \longrightarrow \forall x.(x \in A \Rightarrow x \in B)$$

$$\dfrac{\dfrac{\vdash t \in A \qquad t \in B \vdash}{t \in A \Rightarrow t \in B \vdash} \Rightarrow\text{-L}}{\forall x.(x \in A \Rightarrow x \in B) \vdash} \forall\text{-L}$$

Building the rules

$$\subseteq_{def}: \ A \subseteq B \rightarrow \forall x.(x \in A \Rightarrow x \in B)$$

$$\forall\text{-L}\ \dfrac{\Rightarrow\text{-L}\ \dfrac{\vdash t \in A \qquad t \in B \vdash}{t \in A \Rightarrow t \in B \vdash}}{\forall x.(x \in A \Rightarrow x \in B) \vdash}$$

$$\downarrow$$

$$\subseteq_{def}\text{-L}\ \dfrac{\Gamma \vdash_{+} t \in A, \Delta \qquad \Gamma, t \in B \vdash_{+} \Delta}{\Gamma, A \subseteq B \vdash_{+} \Delta}$$

Permutability problem, eigen variables

$$\dfrac{\dfrac{\dfrac{}{P(x_0) \vdash P(x_0)}\ \text{Ax}}{\forall x.P(x) \vdash P(x_0)}\ \forall\text{-L}}{\forall x.P(x) \vdash \forall x.P(x)}\ \forall\text{-R} \qquad \dfrac{P(t) \vdash \forall x.P(x)}{\forall x.P(x) \vdash \forall x.P(x)}\ \forall\text{-L}$$

- Problems : $\forall$-R then $\forall$-L or $\exists$-R, etc.

Permutability problem, eigen variables

$$\cfrac{\cfrac{\cfrac{}{P(x_0) \vdash P(x_0)}\text{Ax}}{\forall x.P(x) \vdash P(x_0)}\forall\text{-L}}{\forall x.P(x) \vdash \forall x.P(x)}\forall\text{-R} \qquad\qquad \cfrac{P(t) \vdash \forall x.P(x)}{\forall x.P(x) \vdash \forall x.P(x)}\forall\text{-L}$$

- Problems : $\forall$-R then $\forall$-L or $\exists$-R, etc.
- Solution : *focussing*

$$R : P \rightarrow \forall x.(\forall y.A(x,y) \Rightarrow B(x))$$

$$\downarrow$$

$$\begin{array}{rcrcl} R & : & P & \rightarrow & \forall x.(Q(x) \Rightarrow B(x)) \\ R_1 & : & Q(x) & \rightarrow & \forall y.A(x,y) \end{array}$$

Application : arithmetic

- Natural numbers definition $\rightarrow$ induction principle
- "cleaning" the rule:

$$\in_{\mathbb{N}}: n \in \mathbb{N} \rightarrow \forall P.(0 \in P \Rightarrow \forall m.(m \in P \Rightarrow S(m) \in P) \Rightarrow n \in P)$$

$$\downarrow$$

$$\begin{array}{rcccl} \in_{\mathbb{N}} & : & n \in \mathbb{N} & \rightarrow & \forall P.(0 \in P \Rightarrow H(P) \Rightarrow n \in P) \\ hered & : & H(P) & \rightarrow & \forall m.(m \in P \Rightarrow S(m) \in P) \end{array}$$

Application : arithmetic

New deduction rules for *hered*:

$$\textit{hered}\text{-L}\ \frac{\Gamma \vdash_+ m \in P, \Delta \qquad \Gamma, S(m) \in P \vdash_+ \Delta}{\Gamma, H(P) \vdash_+ \Delta}$$

$$\textit{hered}\text{-R}\ \frac{\Gamma, m \in P \vdash_+ S(m) \in P, \Delta}{\Gamma \vdash_+ H(P), \Delta}\ m \notin \mathcal{FV}(\Gamma, \Delta)$$

Application : arithmetic

New deduction rules for *hered*:

$$\textit{hered}\text{-L}\ \frac{\Gamma \vdash_{+} m \in P, \Delta \qquad \Gamma, S(m) \in P \vdash_{+} \Delta}{\Gamma, H(P) \vdash_{+} \Delta}$$

$$\textit{hered}\text{-R}\ \frac{\Gamma, m \in P \vdash_{+} S(m) \in P, \Delta}{\Gamma \vdash_{+} H(P), \Delta}\ m \notin \mathcal{FV}(\Gamma, \Delta)$$

"if from P(n) we can deduce P(n+1) then P is hereditary"

Application : arithmetic

New deduction rules fo $\in_{\mathbb{N}}$:

$$\in_{\mathbb{N}}\text{-L}\ \frac{\Gamma \vdash_+ 0 \in P, \Delta \qquad \Gamma \vdash_+ H(P), \Delta \qquad \Gamma, n \in P \vdash_+ \Delta}{\Gamma, n \in \mathbb{N} \vdash_+ \Delta}$$

$$\in_{\mathbb{N}}\text{-R}\ \frac{0 \in P, H(P) \vdash_+ n \in P, \Delta}{\Gamma \vdash_+ n \in \mathbb{N}, \Delta}\ P \notin \mathcal{FV}(\Gamma, \Delta)$$

Application : arithmetic

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Induction principle

Metaproperties

Theorem (Soundness and completeness of superdeduction)

Every proof $\vdash_+ \phi$ in super sequent calculus can be translated into a proof of $\mathcal{T}h \vdash \phi$ in sequent calculus (soundness) and conversely (completeness).

Proofterms for superdeduction: why?

- replayable traces of proofs
- easily communicable (with other proof-assistants. . .)
- convenient objects to study proof-transformation procedures and especially cut-elimination
- allowing program-extraction through the Curry-Howard-de Bruijn correspondence

Proofterms for classical sequent calculus

Two recent propositions for proofterms for classical sequent calculus

- the $\bar{\lambda}\mu\tilde{\mu}$-calculus (Herbelin 1995)
- Christian Urban's terms (Urban 2000)

Which to choose?

$$\dfrac{\dfrac{\vee\text{-L}\ \dfrac{\varphi_1, \varphi_3 \vdash \varphi_4 \qquad \varphi_2, \varphi_3 \vdash \varphi_4}{\varphi_1 \vee \varphi_2, \varphi_3 \vdash \varphi_4}}{\varphi_1 \vee \varphi_2, \varphi_3 \vdash \varphi_4}\ \textsc{Focus}}{\varphi_1 \vee \varphi_2 \vdash \varphi_3 \Rightarrow \varphi_4}\ \Rightarrow\text{-R}$$

- *Focus* steps are always explicit in $\bar{\lambda}\mu\tilde{\mu}$-calculus.
- They are implicit in superdeduction custom rules !
- They are implicit in Urban's terms. . .

Urban's terms for classical sequent calculus

- Adapted to superdeduction
- Strong Normalisation
- Capturing a large scale of cut elimination procedures

$$M \rhd (x_1 : A_1, \dots x_n : A_n \vdash a_1 : B_1, \dots, a_p : B_p)$$

Implication fragment

$$\textsc{Ax}\ \frac{}{\mathsf{Ax}(x, a) \rhd \Gamma, x : A \vdash a : A, \Delta}$$

$$\textsc{Cut}\ \frac{M_1 \rhd \Gamma \vdash a : A, \Delta \qquad M_2 \rhd \Gamma, x : A \vdash \Delta}{\mathsf{Cut}(\widehat{a}M_1, \widehat{x}M_2) \rhd \Gamma \vdash \Delta}$$

$$\Rightarrow\text{-R}\ \frac{M \rhd \Gamma, x : A \vdash a : B, \Delta}{\mathsf{Imp}_L(\widehat{x}\widehat{a}M, b) \rhd \Gamma \vdash b : A \Rightarrow B, \Delta}$$

$$\Rightarrow\text{-L}\ \frac{M_1 \rhd \Gamma \vdash a : A, \Delta \qquad M_2 \rhd \Gamma, x : B \vdash \Delta}{\mathsf{Imp}_R(\widehat{a}M_1, \widehat{x}M_2, y) \rhd \Gamma, y : A \Rightarrow B \vdash \Delta}$$

$$\textsc{Cut}\;\dfrac{\Rightarrow\text{-R}\;\dfrac{\dfrac{(M)}{\Gamma, A \vdash B, \Delta}}{\Gamma \vdash A \Rightarrow B, \Delta} \qquad \Rightarrow\text{-L}\;\dfrac{\dfrac{(N)}{\Gamma \vdash A, \Delta} \qquad \dfrac{(P)}{\Gamma, B \vdash \Delta}}{\Gamma, A \Rightarrow B \vdash \Delta}}{\Gamma \vdash \Delta}$$

$$\mathsf{Cut}(\widehat{b}\mathsf{Imp}_L(\widehat{x}\widehat{a}M, b), \widehat{z}\mathsf{Imp}_R(\widehat{c}N, \widehat{y}P, z)) \longrightarrow \begin{cases} \color{blue}{\mathsf{Cut}}(\widehat{a}\color{blue}{\mathsf{Cut}}(\widehat{c}N, \widehat{x}M), \widehat{y}P) \\ \color{red}{\mathsf{Cut}}(\widehat{c}N, \widehat{x}\color{red}{\mathsf{Cut}}(\widehat{a}M, \widehat{y}P)) \end{cases}$$

$$\color{blue}{\textsc{Cut}}\;\dfrac{\color{blue}{\textsc{Cut}}\;\dfrac{\dfrac{(N)}{\Gamma \vdash A, \Delta} \qquad \dfrac{(M)}{\Gamma, A \vdash B, \Delta}}{\Gamma \vdash B, \Delta} \qquad \dfrac{(P)}{\Gamma, B \vdash \Delta}}{\Gamma \vdash \Delta}$$

$$\color{red}{\textsc{Cut}}\;\dfrac{\dfrac{(N)}{\Gamma, A \vdash \Delta} \qquad \color{red}{\textsc{Cut}}\;\dfrac{\dfrac{(M)}{\Gamma, A \vdash B, \Delta} \qquad \dfrac{(P)}{\Gamma, B \vdash \Delta}}{\Gamma, A \vdash \Delta}}{\Gamma \vdash \Delta}$$

Extended terms for superdeduction

$$\mathcal{R}\text{-R}\ \frac{\left(\quad \Gamma, \quad A_1^i, \ldots, \quad A_{n_i}^i \vdash \quad : B_1^i, \ldots, \quad B_{p_i}^i, \Delta\right)_i}{\Gamma \vdash a : P, \Delta}$$

$$\mathcal{R}\text{-L}\ \frac{\left(\quad \Gamma, \quad C_1^j, \ldots, \quad : C_{n_j}^j \vdash \quad D_1^i, \ldots, \quad : D_{p_j}^j, \Delta\right)_j}{\Gamma, x : P \vdash \Delta}$$

Extended terms for superdeduction

$$\mathcal{R}\text{-R}\ \frac{\left(M_i \triangleright \Gamma, x_1^i : A_1^i, \ldots, x_{n_i}^i : A_{n_i}^i \vdash a_1^i : B_1^i, \ldots, a_{p_i}^i : B_{p_i}^i, \Delta\right)_i}{\Gamma \vdash a : P, \Delta}$$

$$\mathcal{R}\text{-L}\ \frac{\left(N_j \triangleright \Gamma, y_1^j : C_1^j, \ldots, y_{n_j}^j : C_{n_j}^j \vdash b_1^j : D_1^i, \ldots, b_{p_j}^j : D_{p_j}^j, \Delta\right)_j}{\Gamma, x : P \vdash \Delta}$$

Extended terms for superdeduction

$$\mathcal{R}\text{-R}\ \frac{\left(M_i \triangleright \Gamma, x^i_1 : A^i_1, \ldots, x^i_{n_i} : A^i_{n_i} \vdash a^i_1 : B^i_1, \ldots, a^i_{p_i} : B^i_{p_i}, \Delta\right)_i}{\mathcal{R}_L\left(\left(\widehat{x^i_1} \ldots \widehat{a^i_{p_i}}\right)_i M_i, a\right) \triangleright \Gamma \vdash a : P, \Delta}$$

$$\mathcal{R}\text{-L}\ \frac{\left(N_j \triangleright \Gamma, y^j_1 : C^j_1, \ldots, y^j_{n_j} : C^j_{n_j} \vdash b^j_1 : D^i_1, \ldots, b^j_{p_j} : D^j_{p_j}, \Delta\right)_j}{\mathcal{R}_R\left(\left(\widehat{y^j_1} \ldots \widehat{y^j_{p_j}}\right)_j N_j, x\right) \triangleright \Gamma, x : P \vdash \Delta}$$

Super-cut-elimination

$$\text{Cut}\left(\widehat{a}\mathcal{R}_L\left(\left(\widehat{x_1^i}\ldots\widehat{a_{p_i}^i}\right)_i M_i, a\right), \widehat{x}\mathcal{R}_R\left(\left(\left(\widehat{y_1^j}\ldots\widehat{y_{p_j}^j}\right)_j N_j, x\right)\right)\right) \rightarrow \ ?$$

Super-cut-elimination: an example

$$\mathcal{R} \ : \ A \rightarrow B \wedge \neg A$$

$$\mathcal{R}\text{-R} \ \frac{\Gamma \vdash B, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash A, \Delta}$$

$$\mathcal{R}\text{-L} \ \frac{\Gamma, B \vdash A, \Delta}{\Gamma, A \vdash \Delta}$$

Super-cut-elimination: an example

$$\mathcal{R} \;:\; A \rightarrow B \wedge \neg A$$

$$\mathcal{R}\text{-R}\ \frac{\Gamma \vdash B, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash A, \Delta} \quad \text{stands for} \quad \wedge\text{-R}\ \frac{\Gamma \vdash B, \Delta \qquad \neg\text{-R}\ \dfrac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}}{\Gamma \vdash B \wedge \neg A, \Delta}$$

$$\mathcal{R}\text{-L}\ \frac{\Gamma, B \vdash A, \Delta}{\Gamma, A \vdash \Delta} \quad \text{stands for} \quad \wedge\text{-L}\ \frac{\neg\text{-L}\ \dfrac{\Gamma, B \vdash A, \Delta}{\Gamma, B, \neg A \vdash \Delta}}{\Gamma, B \wedge \neg A \vdash \Delta}$$

Super-cut-elimination: an example

$\mathcal{R}_R(\widehat{b}M_1, \widehat{x}M_2, a)$ stands for $\text{And}_R(\widehat{b}M_1, \widehat{c}\text{Not}_R(\widehat{x}M_2, c), a)$

$\mathcal{R}_L(\widehat{y}\widehat{a}M, x)$ stands for $\text{And}_L(\widehat{y}\widehat{z}\text{Not}_L(\widehat{a}M, z), x)$

$\text{Cut}(\widehat{a}\mathcal{R}_R(\widehat{b}M_1, \widehat{x}M_2, a), \widehat{x}\mathcal{R}_L(\widehat{y}\widehat{a}M, x))$
stands for
$\text{Cut}(\widehat{a}\text{And}_R(\widehat{b}M_1, \widehat{c}\text{Not}_R(\widehat{x}M_2, c), a), \widehat{x}\text{And}_L(\widehat{y}\widehat{z}\text{Not}_L(\widehat{a}M, z), x))$

Super-cut-elimination: an example

$\mathcal{R}_R(\widehat{b}M_1, \widehat{x}M_2, a)$ stands for $\mathrm{And}_R(\widehat{b}M_1, \widehat{c}\mathrm{Not}_R(\widehat{x}M_2, c), a)$

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stands for
$\mathrm{Cut}(\widehat{a}\mathrm{And}_R(\widehat{b}M_1, \widehat{c}\mathrm{Not}_R(\widehat{x}M_2, c), a), \widehat{x}\mathrm{And}_L(\widehat{y}\widehat{z}\mathrm{Not}_L(\widehat{a}M, z), x))$

$\rightarrow \mathrm{Cut}(\widehat{b}M_1, \widehat{y}\mathrm{Cut}(\widehat{c}\mathrm{Not}_R(\widehat{x}M_2, c), \widehat{z}\mathrm{Not}_L(\widehat{a}M, z)))$

$\rightarrow \mathrm{Cut}(\widehat{b}M_1, \widehat{y}\mathrm{Cut}(\widehat{a}M, \widehat{x}M_2))$

Super-cut-elimination: an example

$\mathcal{R}_R(\widehat{b}M_1, \widehat{x}M_2, a)$ stands for $\mathsf{And}_R(\widehat{b}M_1, \widehat{c}\mathsf{Not}_R(\widehat{x}M_2, c), a)$

$\mathcal{R}_L(\widehat{y}\widehat{a}M, x)$ stands for $\mathsf{And}_L(\widehat{y}\widehat{z}\mathsf{Not}_L(\widehat{a}M, z), x)$

$\longrightarrow \mathsf{Cut}(\widehat{b}M_1, \widehat{y}\mathsf{Cut}(\widehat{a}M, \widehat{x}M_2))$

$\mathsf{Cut}(\widehat{a}\mathcal{R}_R(\widehat{b}M_1, \widehat{x}M_2, a), \widehat{x}\mathcal{R}_L(\widehat{y}\widehat{a}M, x))$

stands for

$\mathsf{Cut}(\widehat{a}\mathsf{And}_R(\widehat{b}M_1, \widehat{c}\mathsf{Not}_R(\widehat{x}M_2, c), a), \widehat{x}\mathsf{And}_L(\widehat{y}\widehat{z}\mathsf{Not}_L(\widehat{a}M, z), x))$

$\longrightarrow \mathsf{Cut}(\widehat{b}M_1, \widehat{y}\mathsf{Cut}(\widehat{c}\mathsf{Not}_R(\widehat{x}M_2, c), \widehat{z}\mathsf{Not}_L(\widehat{a}M, z)))$

$\longrightarrow \mathsf{Cut}(\widehat{b}M_1, \widehat{y}\mathsf{Cut}(\widehat{a}M, \widehat{x}M_2))$

Hypothesis for SN

Hypothesis

For a set of proposition rewrite rules $\mathcal{R}$ and for each of its rule $R : P \rightarrow \varphi$:

- *the rewrite relation associated with $\mathcal{R}$ is weakly normalising and confluent;*
- *P contains only first-order variables (no function or constant);*
- *$\mathcal{FV}(\varphi) \subseteq \mathcal{FV}(P)$.*

Strong normalisation result

Theorem (Strong Normalisation)

If the set of proposition rewrite rules satisfies the hypothesis, then the super-cut-elimination is strongly normalising on well-typed extended terms.

Corollary

If the set of proposition rewrite rule satisfies the hypothesis, the underlying first-order theory is consistent.

What's next?

- development of the prototype Lemuridae
  http://rho.loria.fr/lemuridae.html
- interaction with *standard* proof-assistants (Coq, Isabelle. . .)
- relating superdeduction to deduction modulo (*e.g.* concerning cut-elimination), switching to superdeduction modulo
- extend to dependent types, inductive definitions, deep inference

Prototype

Lemuridae : a proof assistant for superdeduction

- Rewrite rules on terms and propositions
- Proof building in the extendible sequent calculus
- Interactive matching rules presentation
- Basic automatic tactics
- Tiny proofchecker

Lemu's "kernel"

$$\wedge\text{-R}\ \frac{\Gamma \vdash \Delta_1, A, \Delta_2 \qquad \Gamma \vdash \Delta_1, B, \Delta_2}{\Gamma \vdash \Delta_1, \underline{A \wedge B}, \Delta_2}$$

```
rule(
  andRightInfo[],
  ( p1@rule(_,_,sequent(g,(d1*,A,d2*)),_),
    p2@rule(_,_,sequent(g,(d1*,B,d2*)),_)
  ),
  sequent(g,(d1*,a,d2*)),
  a@and(A,B)
)
```

Lemu's "kernel"

$$\wedge\text{-R}\ \frac{\Gamma \vdash \Delta_1, A, \Delta_2 \qquad \Gamma \vdash \Delta_1, B, \Delta_2}{\Gamma \vdash \Delta_1, \underline{A \wedge B}, \Delta_2}$$

```
rule(
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rule(
  andRightInfo[],
  ( p1@rule(_,_,sequent(g,(d1*,A,d2*)),_),
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```

Lemu's "kernel"

$$\wedge\text{-R}\ \frac{\Gamma \vdash \Delta_1, A, \Delta_2 \qquad \Gamma \vdash \Delta_1, B, \Delta_2}{\Gamma \vdash \Delta_1, \underline{A \wedge B}, \Delta_2}$$

```
rule(
  andRightInfo[],
  ( p1@rule(_,_,sequent(g,(d1*,A,d2*)),_),
    p2@rule(_,_,sequent(g,(d1*,B,d2*)),_)
  ),
  sequent(g,(d1*,a,d2*)),
  a@and(A,B)
)
```

Lemu's "kernel"

$$\wedge\text{-R}\ \frac{\Gamma \vdash \Delta_1, A, \Delta_2 \qquad \Gamma \vdash \Delta_1, B, \Delta_2}{\Gamma \vdash \Delta_1, \underline{A \wedge B}, \Delta_2}$$

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rule(
  andRightInfo[],
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  andRightInfo[],
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  ( p1@rule(_,_,sequent(g,(d1*,A,d2*)),_),
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Lemu's "kernel"

$$\wedge\text{-R}\ \frac{\Gamma \vdash \Delta_1, A, \Delta_2 \qquad \Gamma \vdash \Delta_1, B, \Delta_2}{\Gamma \vdash \Delta_1, \underline{A \wedge B}, \Delta_2}$$

```
rule(
  andRightInfo[],
  ( p1@rule(_,_,sequent(g,(d1*,A,d2*)),_),
    p2@rule(_,_,sequent(g,(d1*,B,d2*)),_)
  ),
  sequent(g,(d1*,a,d2*)),
  a@and(A,B)
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```

Lemu's "kernel"

$$\wedge\text{-R}\ \frac{\Gamma \vdash \Delta_1, A, \Delta_2 \qquad \Gamma \vdash \Delta_1, B, \Delta_2}{\Gamma \vdash \Delta_1, \underline{A \wedge B}, \Delta_2}$$

```
rule(
  andRightInfo[],
  ( p1@rule(_,_,sequent(g,(d1*,A,d2*)),_),
    p2@rule(_,_,sequent(g,(d1*,B,d2*)),_)
  ),
  sequent(g,(d1*,a,d2*)),
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```

Lemu's "kernel"

$$\wedge\text{-R}\ \frac{\Gamma \vdash \Delta_1, A, \Delta_2 \qquad \Gamma \vdash \Delta_1, B, \Delta_2}{\Gamma \vdash \Delta_1, \underline{A \wedge B}, \Delta_2}$$

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  andRightInfo[],
  ( p1@rule(_,_,sequent(g,(d1*,A,d2*)),_),
    p2@rule(_,_,sequent(g,(d1*,B,d2*)),_)
  ),
  sequent(g,(d1*,a,d2*)),
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```

Lemu's "kernel"

$$\wedge\text{-R}\ \frac{\Gamma \vdash \Delta_1, A, \Delta_2 \qquad \Gamma \vdash \Delta_1, B, \Delta_2}{\Gamma \vdash \Delta_1, \underline{A \wedge B}, \Delta_2}$$

```
rule(
  andRightInfo[],
  ( p1@rule(_,_,sequent(g,(d1*,A,d2*)),_),
    p2@rule(_,_,sequent(g,(d1*,B,d2*)),_)
  ),
  sequent(g,(d1*,a,d2*)),
  a@and(A,B)
)
```

Lemu's "kernel"

$$\wedge\text{-R}\ \frac{\dfrac{\color{red}{p_1}}{\Gamma \vdash \Delta_1, A, \Delta_2} \qquad \dfrac{\color{red}{p_2}}{\Gamma \vdash \Delta_1, B, \Delta_2}}{\Gamma \vdash \Delta_1, \underline{A \wedge B}, \Delta_2}$$

```
rule(
  andRightInfo[],
  ( p1@rule(_,_,sequent(g,(d1*,A,d2*)),_),
    p2@rule(_,_,sequent(g,(d1*,B,d2*)),_)
  ),
  sequent(g,(d1*,a,d2*)),
  a@and(A,B)
)
-> { return (proofcheck(`p1) && proofcheck(`p2)); }
```