Edifices: Böhm Trees for the Symmetric Interaction Combinators

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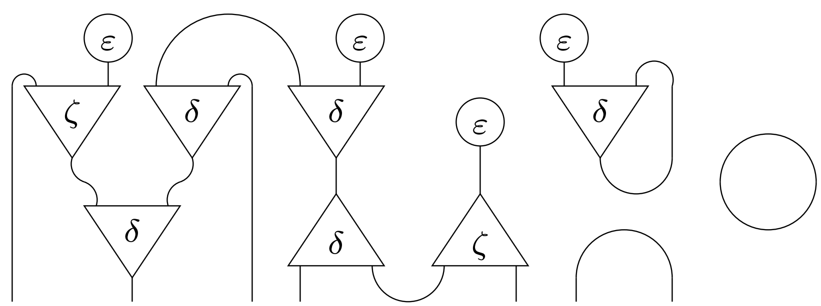
The Symmetric Combinators (Lafont, 1995)

- An extension of untyped unit-free $\mathbf{MLL}$ proof-structures.

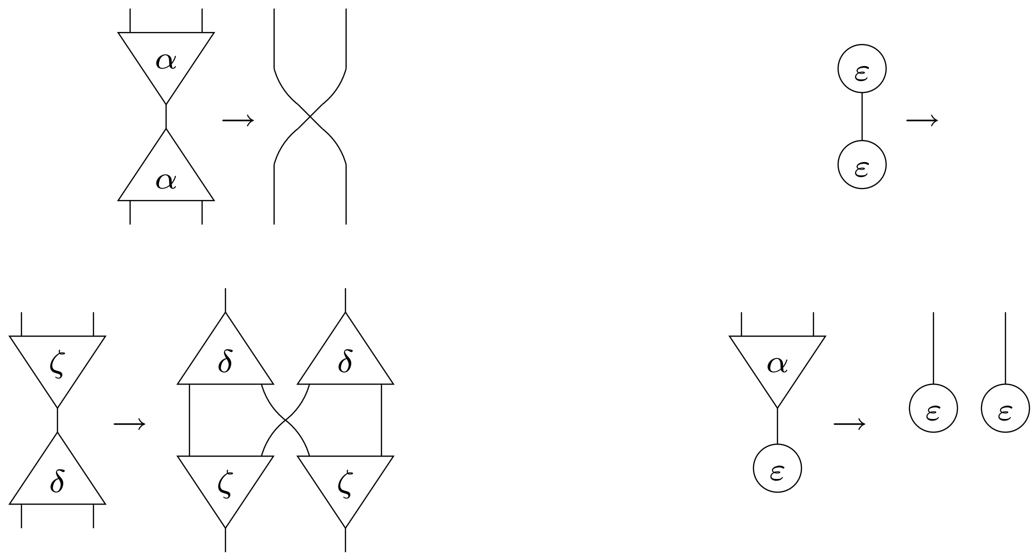
- Like $\mathbf{MLL}$ cut-elimination steps, computational steps are local and asynchronous, but unlike $\mathbf{MLL}$ the symmetric combinators are Turing-complete (in a sense, they are "parallel Turing machines").

- There are two binary combinators (δ and ζ) and a nullary combinator (ε). A *cell* is an occurrence of a combinator; *nets* are made of cells and *wires*, and have a certain number of *free ports*:



Reduction and β-equivalence

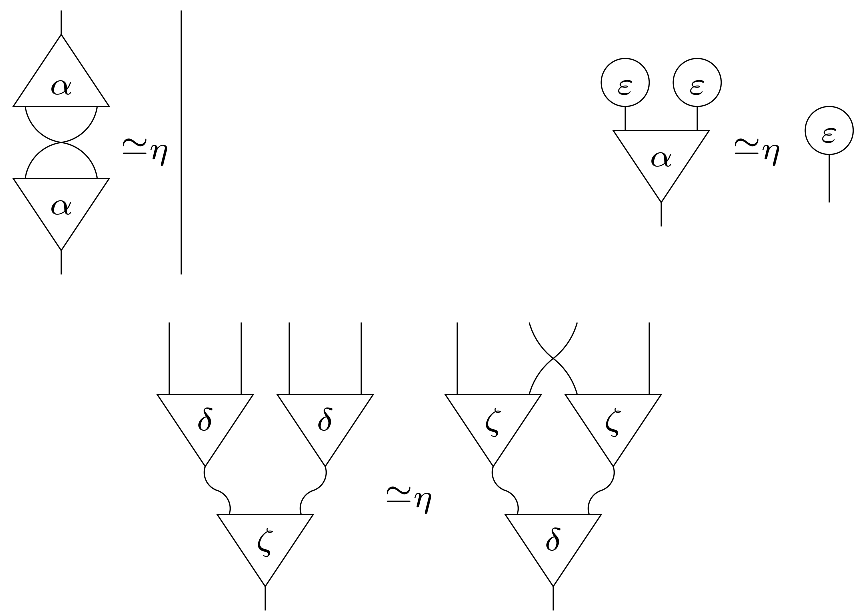
- Two cells connected through their principal ports form an *active pair*. Active pairs can be reduced using the following rules ($\alpha \in \{\delta, \zeta\}$):



- $\mu \simeq_\beta \nu$ iff there exists o such that $\mu \rightarrow^* o$ and $\nu \rightarrow^* o$. Reduction is *strongly confluent*: $\simeq_\beta$ is an equivalence relation (indeed a congruence), and computations are essentially unique.

η-equivalence and $\beta\eta$-equivalence

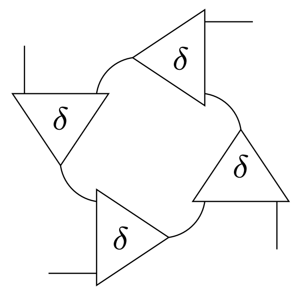
- η-equivalence is defined as the contextual, reflexive, transitive closure of the following equations (where $\alpha \in \{\delta, \zeta\}$):



- As usual, we put $\simeq_{\beta\eta} = (\simeq_\beta \cup \simeq_\eta)^*$.

Internal separation

A *vicious circle* is a cyclic configuration like the following:



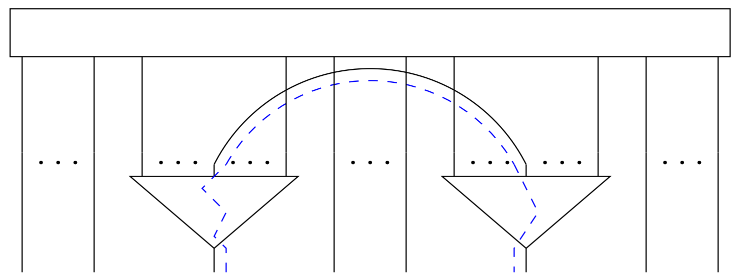
A net is *cut-free* iff it contains no active pairs and no vicious circles. A net is *total* if it reduces to a cut-free net.

Theorem 1. [Mazza, 2006] *Let μ, ν be two total nets such that $\mu \not\simeq_{\beta\eta} \nu$. Then, there exists a cut-free context C such that*

or vice versa.

Observable paths

- We call a path like the following one *observable*:



- The fundamental property of observable paths is that they are stable under reduction. We write $\mu\downarrow$ iff μ contains an observable path.

Observational equivalence

- We deem a net μ *observable*, and we write $\mu\Downarrow$, iff $\mu \to^* \mu'\downarrow$.

- An observable net is like a λ-term in head normal form, but no *principal* hnf can be defined (λ-terms are "intuitionistic", nets are "classical").

- Observational equivalence: $\mu \simeq \nu$, iff, $\forall C$, $C[\mu]\Downarrow \Leftrightarrow C[\nu]\Downarrow$.

- One can prove that $\mu \simeq_{\beta\eta} \nu$ implies $\mu \simeq \nu$. Hence, by separation, $\simeq_{\beta\eta}$ and $\simeq$ coincide on total nets.

Pillars

- Let $\mathcal{C} = \{\mathbf{p}, \mathbf{q}\}^{\mathbb{N}}$, i.e., "the" Cantor set, endowed with the Cantor topology. The elements of $\mathcal{C}$ are ranged over by x, y. We remind that $\mathcal{C}$ is completely metrizable, with distance $d_{\mathcal{C}}(x, y) = 2^{-k}$, where k is the length of the longest common prefix of x, y.

- Let $I \subseteq \mathbb{N}$, and let $\mathcal{P}_I = \mathcal{C} \times \mathcal{C} \times I$. A *pillar* is an element of $\mathcal{P} = \mathcal{P}_{\mathbb{N}}$, ranged over by ξ, υ and denoted by $x \otimes y \,@\, i$. The integer appearing in ξ is the *base* of the pillar, denoted by $\mathsf{b}(\xi)$.

- If we put the discrete topology on $\mathbb{N}$, $\mathcal{P}$ can be endowed with the product topology, which is also metrizable with the distance

$$d(x \otimes y \,@\, i, x' \otimes y' \,@\, i') = \max\{d_{\mathcal{C}}(x, x'), d_{\mathcal{C}}(y, y'), d_{\text{disc}}(i, i')\},$$

where $d_{\text{disc}}(i, i') = 0$ if $i = i'$ and $d_{\text{disc}}(i, i') = 2$ if $i \neq i'$.

Arches

- We denote by $\mathcal{A}_I$ the set of unordered pairs of pillars based at I, i.e., $\mathcal{A}_I = \mathcal{P}_I \times \mathcal{P}_I/\sim$, where $(\xi, \upsilon) \sim (\xi', \upsilon')$ iff $\xi' = \upsilon$ and $\upsilon' = \xi$, or $\xi' = \xi$ and $\upsilon' = \upsilon$.

- An *arch* is an element of $\mathcal{A} = \mathcal{A}_{\mathbb{N}}$, ranged over by $\mathfrak{a}$, and denoted by $\xi \frown \upsilon$ (which by definition is the same as $\upsilon \frown \xi$).

- $\mathcal{P} \times \mathcal{P}$ can be endowed with the product topology, and $\mathcal{A}$ with the quotient topology. This turns out to be metrizable: if $\mathfrak{a} = \xi \frown \upsilon$ and $\mathfrak{a}' = \xi' \frown \upsilon'$, a distance inducing the topology is

$$D(\mathfrak{a}, \mathfrak{a}') = \min\{\max\{d(\xi, \xi'), d(\upsilon, \upsilon')\}, \max\{d(\xi, \upsilon'), d(\upsilon, \xi')\}\}.$$

Edifices

- The space $\mathcal{A}$ is not compact. Indeed, we can give a characterization of its compact subsets:

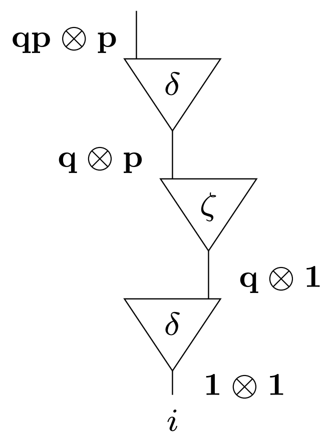
  Proposition 1. *A set $\mathfrak{E} \subseteq \mathcal{A}$ is compact iff it is a closed subset of $\mathcal{A}_I$ for some finite I.*

- An *edifice* is a compact set of arches.

- Note that, by the above proposition, there is identity between closed, compact, and complete (with respect to the metric D) subsets of $\mathcal{A}_I$, whenever I is finite.

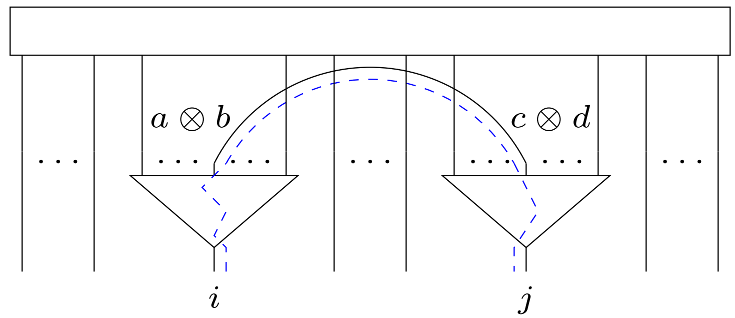
Observable paths as edifices

- A branch of a tree of cells can be identified by an *address* of the form $(a \otimes b, i) \in \{\mathbf{p}, \mathbf{q}\}^* \times \{\mathbf{p}, \mathbf{q}\}^* \times \mathbb{N}$ (this is related to the GoI):



$(\mathbf{qp} \otimes \mathbf{p}, i)$

- Remember that an observable path ϕ is a connection between two branches, of generic addresses $(a \otimes b, i)$ and $(c \otimes d, j)$:



- Then, its edifice is defined as

$$\phi^\bullet = \{ax \otimes by \,@\, i \frown cx \otimes dy \,@\, j \ ; \ \forall x, y \in \mathcal{C}\}.$$

$\phi^\bullet$ is easily seen to be closed. The uniform completion of the addresses reminds of relational semantics (a "locative diagonal"), *copy-cat* strategies, *faxes* in ludics. . .

Nets as edifices

- If μ is a net, and ϕ ranges over all observable paths appearing in all reducts of μ, we define the *pre-edifice* of μ as

$$\mathfrak{E}_0(\mu) = \bigcup \phi^{\bullet}.$$

  We have that $\mathfrak{E}_0(\mu) \subseteq \mathcal{A}_{\{1,\ldots,n\}}$, where n is the number of free ports of μ. Hence, by Proposition 1, the pre-edifice of a normalizable (or, in particular, total) net is an edifice.

- The edifice of a net μ, denoted by $\mathfrak{E}(\mu)$, is the closure of its pre-edifice:

$$\mathfrak{E}(\mu) = \overline{\mathfrak{E}_0(\mu)}.$$

Full abstraction

- The edifice of a net is the analogue of the Nakajima tree of a λ-term (Nakajima, 1975):

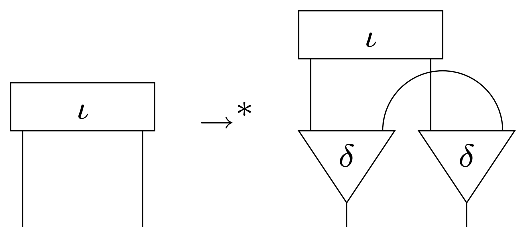
  Theorem 2. [Full abstraction] $\mu \simeq \nu$ *iff* $\mathfrak{E}(\mu) = \mathfrak{E}(\nu)$.

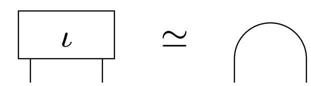
- Compactness (hence completeness) is fundamental: $\mathfrak{E}_0(\cdot)$ gives an adequate, but *not* fully abstract semantics, because of *infinite* η*-reduction* (Wadsworth 1976, Hyland 1976).

- In fact, the phenomenon of infinite η-reduction receives a precise topological interpretation in edifices, which is not given by Böhm or Nakajima trees.

A (hopefully) clarifying example

There exists a net ι such that





$\mathfrak{E}_0(\iota)$ contains, for all $x, y \in \mathcal{C}$, a Cauchy sequence of the form

$$\mathfrak{a}_n = \mathbf{p}^n\mathbf{q}x \otimes y \,@\, 1 \frown \mathbf{p}^n\mathbf{q}x \otimes y \,@\, 2,$$

without containing its limit

$$\mathbf{p}^\infty \otimes y \,@\, 1 \frown \mathbf{p}^\infty \otimes y \,@\, 2.$$

Thanks to the addition of these limits, we obtain $\mathfrak{E}(\iota) = \mathfrak{E}(\frown)$.

Further work

- The set of all edifices is huge (its cardinality is $2^{2^{\aleph_0}}$). Could one restrict the definition in order to obtain a full completeness result?

- A notion of composition can be defined on edifices. This is done by considering the equivalent of plays in games semantics, or of the execution formula in the GoI. Concretely, it involves computing certain sequences of arches.

- Can one define a category out of this? In other words, can one find a *typed* version of the symmetric combinators?

- Does the length of the sequences appearing in the composition of edifices say anything about the *runtime* of nets (like nilpotency in the GoI)?