Proving termination using dependent types: the case of xor-terms

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Outline

Formal models of cryptographic systems

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- Protocols
- Security APIs

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Examples from a security API called CCA
(Common Cryptographic Architecture):

$$x, y, \{z\}_{x \oplus KP \oplus KM} \mapsto \{z \oplus y\}_{x \oplus KP \oplus KM}$$
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Reasoning involves:

Commutativity:	$x \oplus y \simeq y \oplus x$
Associativity:	$(x \oplus y) \oplus z \simeq x \oplus (y \oplus z)$
Neutral element:	$x \oplus 0 \simeq x$
Involutivity:	$x \oplus x \simeq 0$

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General setting: quotiented first order-terms

We are given

- A type of terms $\mathcal{T}$ with constructors C_k:
  Inductive $\mathcal{T}$: Set :=
  $\mid C_1 : \mathcal{T}$
  $\vdots$
  $\mid C_k : \ldots \to \mathcal{T} \ldots \to \mathcal{T} \ldots \to \mathcal{T}$
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- A congruence $\simeq : \mathcal{T} \rightarrow \mathcal{T} \rightarrow \mathit{Prop}$
  - For each constructor C_k
    $\forall a, \ldots x_1, y_1, b, \ldots x_2, y_2, \ldots c,$
    $x_1 \simeq y_1 \rightarrow x_2 \simeq y_2 \rightarrow$
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  - specific laws, e.g. $\forall xy, C_2 \, x \, C_1 \, y \simeq C_2 \, y \, x$

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We want to reason on $\mathcal{T}$ up to $\simeq$

Already well-known examples

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- finite bags represented by finite lists
- algebra of formal arithmetic expressions
  $+$ is associative, commutative, 0 is neutral
  $\times$ is associative, commutative, 1 is neutral
  $\times$ distributes over $+$
- (mobile) process calculi, chemical abstract machines
  parallel composition and choice operators are AC

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- Explicit approach :
  - Define a normalization function N on $\mathcal{T}$
  - Compare terms using syntactic equality on their norms : $x \simeq y$ iff $N\,x = N\,y$

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Cryptographic systems need more

Reasoning on such systems involves

- comparing terms up to AC + involutivity of $\oplus$:

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→ normalization is needed!

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- combination of polynomial and lexicographic ordering
- other approaches (lpo, rpo,. . .): overkill?
- AC matching: a non trivial matter

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Better formulation: $t : \mathcal{T}$ transformed into $t' : \mathcal{T}'$
$\mathcal{T}'$ enriched version of $\mathcal{T}$,
trivial forgetful morphism $\mathcal{T}' \to \mathcal{T}$.

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Interesting part $= \mathcal{T} \rightarrow \mathcal{T}'$

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- normalizing sauce = rearranging + removing duplicates

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$\mathcal{T}$ as a lasagna

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Inductive $\mathcal{T}$: Set :=

$| \; Zero\colon \mathcal{T}$

$| \; PC\colon public\_const \rightarrow \mathcal{T}$ $\qquad | \; SC\colon secret\_const \rightarrow \mathcal{T}$

$| \; E\colon \mathcal{T} \rightarrow \mathcal{T} \rightarrow \mathcal{T}$

$| \; Xor\colon \mathcal{T} \rightarrow \mathcal{T} \rightarrow \mathcal{T}$

$| \; Hash\colon \mathcal{T} \rightarrow \mathcal{T} \rightarrow \mathcal{T}.$

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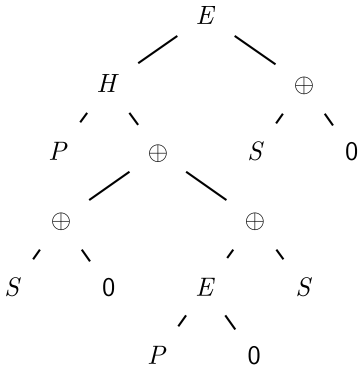

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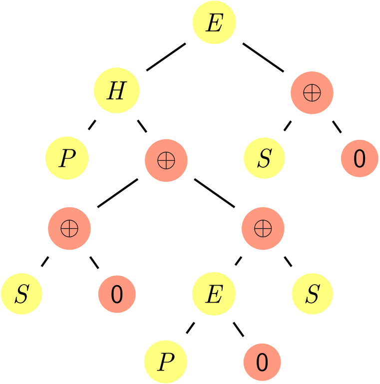

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Decomposing $\mathcal{T}$

Inductive $\mathcal{T}_x$:Set :=

$| \; X\_Zero : \mathcal{T}_x$

$| \; X\_Xor : \mathcal{T}_x \longrightarrow \mathcal{T}_x \longrightarrow \mathcal{T}_x$

Inductive $\mathcal{T}_n$: Set :=

$| \; NX\_PC : public\_const \longrightarrow \mathcal{T}_n$

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Decomposing $\mathcal{T}$

Variable A : Set.

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| $NX\_sum : A \rightarrow \mathcal{T}_n$

Outline

- Motivation
  - The case of cryptographic systems
  - State of the art
  - Back to cryptographic systems
  - Solving strategies
- Solution (intuitive)
  - Basic idea
  - Analyse of $\mathcal{T}$
  - Decomposing $\mathcal{T}$
  - Stratifying and normalizing a term
- Issues
  - Lifting
  - Alternation
  - Forbid fake inclusions
  - Fixpoints
  - Conversion rule
- Conclusion

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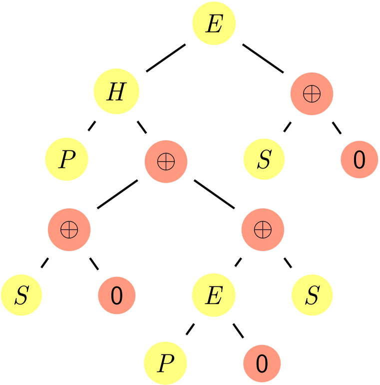

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  - need to lift $\mathcal{L}_x\, n$ and $\mathcal{L}_x\, m$ to $\mathcal{L}_x\,(\max\, n\, m)$

Lifting lasagna

$$\mathcal{L}_x\, k \overset{\text{def}}{=\!=} \underbrace{\mathcal{T}_x(\mathcal{T}_n(\mathcal{T}_x(\dots(\emptyset))))}_{k \text{ layers}} \text{ for } k \text{ large enough.}$$

- What is k?
- The number of layers on the left subterm and on the right subterm are different in general.

Take the max

- Standard solution: {le n m} + {le m n}
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  - $lift_x : \mathcal{L}_x\, k \to \mathcal{L}_x\,(k + d)$, $lift_n : \mathcal{L}_n\, k \to \mathcal{L}_n\,(k + d)$
  - No need to proof that max is the max.

Outline

Internalizing alternation

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Well designed types help us to design programs

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Control them using alternating natural numbers

Inductive alt_{even}: Set :=
 | 0_e: alt_{even}
 | $S_{o\rightarrow e}$: $alt_{odd} \rightarrow alt_{even}$
with alt_{odd}: Set :=
 | $S_{e\rightarrow o}$: $alt_{even} \rightarrow alt_{odd}$

Outline

- Motivation
  - The case of cryptographic systems
  - State of the art
  - Back to cryptographic systems
  - Solving strategies
- Solution (intuitive)
  - Basic idea
  - Analyse of $\mathcal{T}$
  - Decomposing $\mathcal{T}$
  - Stratifying and normalizing a term
- Issues
  - Lifting
  - Alternation
  - Forbid fake inclusions
  - Fixpoints
  - Conversion rule
- Conclusion

Forbid fake inclusions

Forbid fake inclusions

Inductive $\mathcal{T}_x$: Set :=
| X_Zero : $\mathcal{T}_x$
| X_ns : $A \longrightarrow \mathcal{T}_x$
| X_Xor : $\mathcal{T}_x \longrightarrow \mathcal{T}_x \longrightarrow \mathcal{T}_x$

Inductive $\mathcal{T}_n$: Set :=
| NX_PC : $public\_const \longrightarrow \mathcal{T}_n$
| NX_SC : $secret\_const \longrightarrow \mathcal{T}_n$
| NX_sum : $A \longrightarrow \mathcal{T}_n$
| NX_E : $\mathcal{T}_n \longrightarrow \mathcal{T}_n \longrightarrow \mathcal{T}_n$
| NX_Hash : $\mathcal{T}_n \longrightarrow \mathcal{T}_n \longrightarrow \mathcal{T}_n$

Forbid fake inclusions

$$
\begin{array}{l}
\text{Inductive } \mathcal{T}_x\text{:}\quad Set \ := \\
\quad \mid X\_Zero : \quad \mathcal{T}_x \\
\quad \mid X\_ns : \quad A \rightarrow \mathcal{T}_x \\
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\end{array}
$$

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\quad \mid NX\_Hash : \ \mathcal{T}_n \ \rightarrow \mathcal{T}_n \ \rightarrow \mathcal{T}_n
\end{array}
$$

$$X\_ns \ (NX\_sum \ (\ X\_ns \ (NX\_sum \ (\ldots))))$$

Forbid fake inclusions

Inductive $\mathcal{T}_x$: $bool \longrightarrow Set$:=
$\mid X\_Zero : \forall\ b,\ \mathcal{T}_x\ b$
$\mid X\_ns : \forall\ b,\ Is\_true\ b \longrightarrow A \longrightarrow \mathcal{T}_x\ b$
$\mid X\_Xor : \forall\ b,\ \mathcal{T}_x\ true \longrightarrow \mathcal{T}_x\ true \longrightarrow \mathcal{T}_x\ b$

Inductive $\mathcal{T}_n$: $bool \longrightarrow Set$:=
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$X\_ns\ (NX\_sum\ (\ X\_ns\ (NX\_sum\ (\ldots))))$

Outline

Mutual induction

- Prefer fixpoints: built-in computation, no inversion

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Many 10 lines definitions, almost no theorem

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```
Fixpoint $lift\_lasagna\_x$ $e_1$ $e_2$ $\{struct\ e_1\}$ :
   $\mathcal{L}_x\ e_1 \longrightarrow \mathcal{L}_x\ (e_1 + e_2)$ :=
   match $e_1$ return $\mathcal{L}_x\ e_1 \longrightarrow \mathcal{L}_x\ (e_1 + e_2)$ with
   | $0_e$ $\Rightarrow$ fun $emp$ $\Rightarrow$ match $emp$ with end
   | $S_{o \rightarrow e}\ o_1$ $\Rightarrow$ $map_x\ (lift\_lasagna\_n\ o_1\ e_2)\ false$
   end
with $lift\_lasagna\_n$ $o_1$ $e_2$ $\{struct\ o_1\}$ :
   $\mathcal{L}_n\ o_1 \longrightarrow \mathcal{L}_n\ (o_1 + e_2)$ :=
   match $o_1$ return $\mathcal{L}_n\ o_1 \longrightarrow \mathcal{L}_n\ (o_1 + e_2)$ with
   | $S_{e \rightarrow o}\ e_1$ $\Rightarrow$ $map_n\ (lift\_lasagna\_x\ e_1\ e_2)\ false$
   end.
```

Outline

Conversion rule

Conversion rule

Used everywhere

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Used everywhere

$$
\begin{array}{l}
\text{Definition } bin\_xor \\
\quad (bin : \forall\ A\ b,\ \mathcal{T}_x\ A\ true \rightarrow \mathcal{T}_x\ A\ true \rightarrow \mathcal{T}_x\ A\ b)\ o_1\ o_2\ b \\
\quad (l_1 : lasagna\_cand\_x\ o_1\ true) \\
\quad (l_2 : lasagna\_cand\_x\ o_2\ true) : \\
\quad lasagna\_cand\_x\ (max\_oo\ o_1\ o_2)\ b := \\
\\
\quad bin\ (\mathcal{L}_n\ (max\_oo\ o_1\ o_2))\ b \\
\qquad\quad (lift\_lasagna\_cand\_x\ true\ o_1\ (o_2\ \text{-}\ o_1)\ l_1) \\
\qquad\quad (coerce\_max\_comm \\
\qquad\qquad (lift\_lasagna\_cand\_x\ true\ o_2\ (o_1\ \text{-}\ o_2)\ l_2)).
\end{array}
$$

Conclusion

Type theory is flexible

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- Polymorphism

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- Mutually inductive types

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