On the Stability by Union of Reducibility Candidates

Colin Riba

INPL & LORIA
http://loria.fr/~riba/

Journée Déduction Modulo
14 Janvier 2006

Outline

Outline

Our starting point:

- Strong normalization of λ-calculus plus rewriting in presence of union types [Blanqui & Riba 06].

More generally,

- Simple characterization of reducibility candidates and saturated sets.
- Better understanding of reducibility.

Outline

Terms

- Terms:

$$t, u \in \Lambda \quad ::= \quad x \quad | \quad t\,u \quad | \quad \lambda x.t \quad | \quad \pi_i t \quad | \quad \langle t, u \rangle \ .$$

- Reductions:

$$(\lambda x.t)u \mapsto_\beta t[u/x] \qquad \pi_i \langle t_1, t_2 \rangle \mapsto_\beta t_i \ .$$

- Two kinds of values:

$$\lambda x.t \qquad \text{and} \qquad \langle t, u \rangle$$

Types

- Types:

$$T, U \in \mathcal{T} \quad ::= \quad \mathrm{B} \quad | \quad T \Rightarrow U \quad | \quad T \times U$$

- Typing rules:

$$(\mathrm{Ax})\ \frac{}{\Gamma, x : T \ \vdash\ x : T}$$

$$(\Rightarrow \mathrm{I})\ \frac{\Gamma, x : U \ \vdash\ t : T}{\Gamma \ \vdash\ \lambda x.t : U \Rightarrow T} \qquad (\Rightarrow \mathrm{E})\ \frac{\Gamma \ \vdash\ t : U \Rightarrow T \qquad \Gamma \ \vdash\ u : U}{\Gamma \ \vdash\ t\, u : T}$$

$$(\times \mathrm{I})\ \frac{\Gamma \ \vdash\ t_1 : T_1 \qquad \Gamma \ \vdash\ t_2 : T_2}{\Gamma \ \vdash\ \langle t_1, t_2 \rangle : T_1 \times T_2} \qquad (\times \mathrm{E})\ \frac{\Gamma \ \vdash\ t : T_1 \times T_2}{\Gamma \ \vdash\ \pi_i t : T_i}\ (i \in \{1, 2\})$$

Outline

- Interpret types $T \in \mathcal{T}$ as sets of $\mathcal{SN}$ terms $[\![T]\!] \subseteq \mathcal{SN}$.
- Prove the *soundness* of the interpretation:

  If $\Gamma \vdash t : T$ *and* $\sigma(x) \in [\![A]\!]$ *for all* $(x : A) \in \Gamma$*, then* $\sigma(t) \in [\![T]\!]$.

- $[\![T]\!]$ must satisfy some closure conditions.

- Interpret types $T \in \mathcal{T}$ as sets of $\mathcal{SN}$ terms $[\![T]\!] \subseteq \mathcal{SN}$.
- Prove the *soundness* of the interpretation:

  If $\Gamma \vdash t : T$ *and* $\sigma(x) \in [\![A]\!]$ *for all* $(x : A) \in \Gamma$, *then* $\sigma(t) \in [\![T]\!]$.

- $[\![T]\!]$ must satisfy some closure conditions.

Outline

- Arrow:

$$\mathcal{A} \Rightarrow \mathcal{B} \quad =_{\text{def}} \quad \{t \mid \forall u\, (u \in \mathcal{A} \;\Rightarrow\; tu \in \mathcal{B})\}$$

- Product:

$$\mathcal{A} \times \mathcal{B} \quad =_{\text{def}} \quad \{t \mid \pi_1 t \in \mathcal{A} \;\wedge\; \pi_2 t \in \mathcal{B}\}$$

Outline

Neutrality

- Atomic elimination contexts:

$$\epsilon[\ ] \quad ::= \quad [\ ]\,t \quad | \quad \pi_i[\ ]$$

- Elimination contexts: $E[\ ] ::= [\ ] \mid E[\epsilon[\ ]]$.
- t is *neutral* ($t \in \mathcal{N}$) iff t is not a value.
- If $t \in \mathcal{N}$, then
  1. $E[t] \in \mathcal{N}$
  2. If $E[t] \to v$, then $v = E'[t']$ with $(E[\ ], t) \to (E'[\ ], t')$.

Neutrality

- Atomic elimination contexts:

$$\epsilon[\,] \quad ::= \quad [\,]\,t \quad | \quad \pi_i[\,]$$

- Elimination contexts: $E[\,] ::= [\,] \mid E[\epsilon[\,]]$.
- t is *neutral* ($t \in \mathcal{N}$) iff t is not a value.
- If $t \in \mathcal{N}$, then
  1. $E[t] \in \mathcal{N}$
  2. If $E[t] \rightarrow v$, then $v = E'[t']$ with $(E[\,], t) \rightarrow (E'[\,], t')$.

Definitions

- $C \in \mathcal{CR}$ iff $C \subseteq \mathcal{SN}$ and
  $(\mathcal{CR}0)$ if $t \in C$ and $t \to u$ then $u \in C$,
  $(\mathcal{CR}1)$ if $t \in \mathcal{N}$ and $(\forall u\ (t \to u \ \Rightarrow\ u \in C))$ then $t \in C$.
- If $X \subseteq \mathcal{SN}$, $\overline{X}$ is the smallest set such that $X \subseteq \overline{X} \in \mathcal{CR}$.

Application

- Let $X \in \{\Rightarrow, \times\}$

$$\text{If } \mathcal{A}, \mathcal{B} \in \mathcal{CR}, \text{ then } \mathcal{A} \; X \; \mathcal{B} \in \mathcal{CR}.$$

- $\mathcal{A} \; X \; \mathcal{B} \subseteq \mathcal{SN}$.
- $\mathcal{A} \; X \; \mathcal{B}$ stable by reduction.
- Let $t \in \mathcal{N}$ and $(t)_{\rightarrow} \subseteq \mathcal{A} \; X \; \mathcal{B}$.
  Since $\epsilon[t] \in \mathcal{N}$, apply $(\mathcal{CR}1)$ on $\mathcal{A}, \mathcal{B}$.
  By induction on $\epsilon[\;] \in \mathcal{SN}$. Let $(\epsilon[\;], t) \rightarrow v$.
  If $v = \epsilon[t']$ with $t \rightarrow t'$, we conclude by assumption.
  Otherwise, $v = \epsilon'[t]$, and we conclude by induction hypothesis.

Application

- Let $X \in \{\Rightarrow, \times\}$

$$\text{If } \mathcal{A}, \mathcal{B} \in \mathcal{CR}, \text{ then } \mathcal{A}\, X\, \mathcal{B} \in \mathcal{CR}.$$

- $\mathcal{A}\, X\, \mathcal{B} \subseteq \mathcal{SN}$.
- $\mathcal{A}\, X\, \mathcal{B}$ stable by reduction.
- Let $t \in \mathcal{N}$ and $(t)_{\rightarrow} \subseteq \mathcal{A}\, X\, \mathcal{B}$.
  Since $\epsilon[t] \in \mathcal{N}$, apply $(\mathcal{CR}1)$ on $\mathcal{A}, \mathcal{B}$.
  By induction on $\epsilon[\ ] \in \mathcal{SN}$. Let $(\epsilon[\ ], t) \rightarrow v$.
  If $v = \epsilon[t']$ with $t \rightarrow t'$, we conclude by assumption.
  Otherwise, $v = \epsilon'[t]$, and we conclude by induction hypothesis.

Application

- Let $X \in \{\Rightarrow, \times\}$

$$\text{If } \mathcal{A}, \mathcal{B} \in \mathcal{CR}, \text{ then } \mathcal{A}\ X\ \mathcal{B} \in \mathcal{CR}.$$

- $\mathcal{A}\ X\ \mathcal{B} \subseteq \mathcal{SN}$.
- $\mathcal{A}\ X\ \mathcal{B}$ stable by reduction.
- Let $t \in \mathcal{N}$ and $(t)_{\rightarrow} \subseteq \mathcal{A}\ X\ \mathcal{B}$.
  Since $\epsilon[t] \in \mathcal{N}$, apply $(\mathcal{CR}1)$ on $\mathcal{A}, \mathcal{B}$.
  By induction on $\epsilon[\ ] \in \mathcal{SN}$. Let $(\epsilon[\ ], t) \rightarrow v$.
  If $v = \epsilon[t']$ with $t \rightarrow t'$, we conclude by assumption.
  Otherwise, $v = \epsilon'[t]$, and we conclude by induction hypothesis.

Application

- Let $X \in \{\Rightarrow, \times\}$

$$\text{If } \mathcal{A}, \mathcal{B} \in \mathcal{CR}, \text{ then } \mathcal{A}\ X\ \mathcal{B} \in \mathcal{CR}.$$

- $\mathcal{A}\ X\ \mathcal{B} \subseteq \mathcal{SN}$.
- $\mathcal{A}\ X\ \mathcal{B}$ stable by reduction.
- Let $t \in \mathcal{N}$ and $(t)_{\rightarrow} \subseteq \mathcal{A}\ X\ \mathcal{B}$.
  Since $\epsilon[t] \in \mathcal{N}$, apply $(\mathcal{CR}1)$ on $\mathcal{A}, \mathcal{B}$.
  By induction on $\epsilon[\ ] \in \mathcal{SN}$. Let $(\epsilon[\ ], t) \rightarrow v$.
  If $v = \epsilon[t']$ with $t \rightarrow t'$, we conclude by assumption.
  Otherwise, $v = \epsilon'[t]$, and we conclude by induction hypothesis.

Application

- Let $X \in \{\Rightarrow, \times\}$

$$\text{If } \mathcal{A}, \mathcal{B} \in \mathcal{CR}, \text{ then } \mathcal{A}\, X\, \mathcal{B} \in \mathcal{CR}.$$

- $\mathcal{A}\, X\, \mathcal{B} \subseteq \mathcal{SN}$.
- $\mathcal{A}\, X\, \mathcal{B}$ stable by reduction.
- Let $t \in \mathcal{N}$ and $(t)_{\rightarrow} \subseteq \mathcal{A}\, X\, \mathcal{B}$.
  Since $\epsilon[t] \in \mathcal{N}$, apply $(\mathcal{CR}1)$ on $\mathcal{A}, \mathcal{B}$.
  By induction on $\epsilon[\ ] \in \mathcal{SN}$. Let $(\epsilon[\ ], t) \rightarrow v$.
  If $v = \epsilon[t']$ with $t \rightarrow t'$, we conclude by assumption.
  Otherwise, $v = \epsilon'[t]$, and we conclude by induction hypothesis.

Application

- Let $X \in \{\Rightarrow, \times\}$

$$\text{If } \mathcal{A}, \mathcal{B} \in \mathcal{CR}, \text{ then } \mathcal{A}\ X\ \mathcal{B} \in \mathcal{CR}.$$

- $\mathcal{A}\ X\ \mathcal{B} \subseteq \mathcal{SN}$.
- $\mathcal{A}\ X\ \mathcal{B}$ stable by reduction.
- Let $t \in \mathcal{N}$ and $(t)_{\rightarrow} \subseteq \mathcal{A}\ X\ \mathcal{B}$.
  Since $\epsilon[t] \in \mathcal{N}$, apply $(\mathcal{CR}1)$ on $\mathcal{A}, \mathcal{B}$.
  By induction on $\epsilon[\ ] \in \mathcal{SN}$. Let $(\epsilon[\ ], t) \rightarrow v$.
  If $v = \epsilon[t']$ with $t \rightarrow t'$, we conclude by assumption.
  Otherwise, $v = \epsilon'[t]$, and we conclude by induction hypothesis.

Outline

- Let $\mathcal{C} \subseteq \mathcal{CR}$. We want

$$\bigcup \mathcal{C} \quad =_{\text{def}} \quad \bigcup_{C \in \mathcal{C}} C \quad \in \quad \mathcal{CR}$$

- $\mathcal{SN}$ and $(\mathcal{CR}0)$ are OK.
- $(\mathcal{CR}1)$
  - Let $t \in \mathcal{N}$ with $(t)_{\rightarrow} \subseteq \bigcup \mathcal{C}$.
  - We need some $C \in \mathcal{C}$ such that $(t)_{\rightarrow} \in C$.

Outline

- Let $\mathcal{CRU}$ be the smallest set such that $\mathcal{CR} \subseteq \mathcal{CRU}$ and $\mathcal{C} \subseteq \mathcal{CRU} \Rightarrow \bigcup \mathcal{C} \in \mathcal{CRU}$.
- Hence, $\mathcal{CR}$ is stable by union iff $\mathcal{CR} = \mathcal{CRU}$.
- **Theorem 1.** $C \in \mathcal{CRU}$ iff

$$C = \bigcup \{\bar{t} \mid t \in C\}$$

- Note that for all $C \in \mathcal{CR}$,

$$C = \bigcup \{\bar{t} \mid t \in C\}$$

- Let $\mathcal{CRU}$ be the smallest set such that $\mathcal{CR} \subseteq \mathcal{CRU}$ and $\mathcal{C} \subseteq \mathcal{CRU} \;\Rightarrow\; \bigcup \mathcal{C} \in \mathcal{CRU}$.
- Hence, $\mathcal{CR}$ is stable by union iff $\mathcal{CR} = \mathcal{CRU}$.
- **Theorem 1.** $C \in \mathcal{CRU}$ iff

$$C \quad = \quad \bigcup \{\bar{t} \;\mid\; t \in C\}$$

- Note that for all $C \in \mathcal{CR}$,

$$C \quad = \quad \bigcup \{\bar{t} \;\mid\; t \in C\}$$

Outline

Definitions

- Let $t \sqsubseteq u$ iff for all value v,

$$t \rightarrow^* v \quad \Rightarrow \quad u \rightarrow^* v \; .$$

- Let $t \sqsubseteq_{\mathcal{SN}} u$ iff $t \sqsubseteq u$ and $t, u \in \mathcal{SN}$.
- We have $t \sqsubseteq u$ iff for all value v,

$$\forall E[\;]\, (E[t] \rightarrow^* v \quad \Rightarrow \quad E[u] \rightarrow^* v) \; .$$

Definitions

- Let $t \sqsubseteq u$ iff for all value v,
$$t \rightarrow^* v \quad \Rightarrow \quad u \rightarrow^* v \ .$$

- Let $t \sqsubseteq_{\mathcal{SN}} u$ iff $t \sqsubseteq u$ and $t, u \in \mathcal{SN}$.
- We have $t \sqsubseteq u$ iff for all value v,
$$\forall E[\ ]\ (E[t] \rightarrow^* v \quad \Rightarrow \quad E[u] \rightarrow^* v)\ .$$

Results

- **Theorem 2.** $\bar{t} = \{u \mid u \sqsubseteq_{\mathcal{SN}} t\}$.
- **Corollary 1.** $C \in \mathcal{CRU}$ iff

$$C = \{u \mid u \sqsubseteq_{\mathcal{SN}} t \in C\}$$

- **Corollary 2.** $\mathcal{CR}$ is stable by union iff $\mathcal{CR}$ is the set of all $C \subseteq \mathcal{SN}$ such that

$$C = \{u \mid u \sqsubseteq_{\mathcal{SN}} t \in C\}$$

Principal Reduct

- Therefore, $\mathcal{CR} = \mathcal{CRU}$ iff for all C,

$$C = \{u \mid u \sqsubseteq_{\mathcal{SN}} t \in C\} \qquad \Rightarrow \qquad C \in \mathcal{CR}$$

($\mathcal{CR}0$) Since $(u \sqsubseteq_{\mathcal{SN}} t \ \wedge \ u \to u') \ \Rightarrow \ u' \sqsubseteq_{\mathcal{SN}} t$.
($\mathcal{CR}1$) Let $t \in \mathcal{N}$ such that $(t)_{\to} \subseteq C$. We need some $u \in C$ such that $u \sqsubseteq_{\mathcal{SN}} t$.

- **Theorem 3.** $\mathcal{CR} = \mathcal{CRU}$ iff for every $t \in \mathcal{N} \cap \mathcal{SN}$, there is $u \in (t)_{\to}$ such that $t \sqsubseteq u$.

  Note that $u = \max_{\sqsubseteq}(t)_{\to}$.
  We say that u is a *principal reduct* of t.

Principal Reduct

- Therefore, $\mathcal{CR} = \mathcal{CRU}$ iff for all C,

$$C = \{u \mid u \sqsubseteq_{\mathcal{SN}} t \in C\} \qquad \Rightarrow \qquad C \in \mathcal{CR}$$

(CR0) Since $(u \sqsubseteq_{\mathcal{SN}} t \ \wedge \ u \to u') \ \Rightarrow \ u' \sqsubseteq_{\mathcal{SN}} t$.
(CR1) Let $t \in \mathcal{N}$ such that $(t)_{\to} \subseteq C$. We need some $u \in C$ such that $u \sqsubseteq_{\mathcal{SN}} t$.

- **Theorem 3.** $\mathcal{CR} = \mathcal{CRU}$ iff for every $t \in \mathcal{N} \cap \mathcal{SN}$, there is $u \in (t)_{\to}$ such that $t \sqsubseteq u$.

  Note that $u = \max_{\sqsubseteq} (t)_{\to}$.
  We say that u is a *principal reduct* of t.

Outline

In $\boldsymbol{\lambda}_{\Rightarrow\times}$, we seek for principal reducts of $t \in \mathcal{N} \cap \mathcal{SN}$.

Weak Standardization:

- Let $t \rightarrow_\beta u$ and $E[t] \rightarrow v$ with $v \neq E[u]$.
- Then $v = E'[t']$ with $(E[\,], t) \rightarrow (E'[\,], t')$ and there exists u' such that $t' \rightarrow_\beta u'$ and $E[u] \rightarrow^* E'[u']$.

In $\boldsymbol{\lambda}_{\Rightarrow\times}$, we seek for principal reducts of $t \in \mathcal{N} \cap \mathcal{SN}$.

Weak Standardization:

- Let $t \rightarrow_\beta u$ and $E[t] \rightarrow v$ with $v \neq E[u]$.
- Then $v = E'[t']$ with $(E[\,],t) \rightarrow (E'[\,],t')$ and there exists u' such that $t' \rightarrow_\beta u'$ and $E[u] \rightarrow^* E'[u']$.

In a recent paper [Riba 07]:

- We have given a characterization of the stability by union of $\mathcal{CR}$.
- We have shown that it holds for $\lambda_{\Rightarrow\times}$, and for more elaborated calculi.
- It those cases, we have shown that Girard's Reducibility candidates are exactly the Tait's saturated sets that are stable by reduction.

Future Work:

- Application to orthogonal rewriting.
- What happens when mixing union types and non-determinism?