On the Stability by Union of Reducibility Candidates

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Colin Riba On the Stability by Union of Reducibility Candidates

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Outline

- Introduction
 - Motivations
 - The Calculus $\lambda_{\Rightarrow imes imes}$
- 2 Reducibility Candidates
 - General Idea
 - Interpretation of Types
 - Girard's Reducibility Candidates
- Stability by Union
 - Main Point
 - General Considerations
 - Weak Observational Preorder
- 4 Application to $\lambda_{\Rightarrow imes}$
 - Application to λ_{⇒×}

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Motivations The Calculus $\lambda_{\Rightarrow imes}$

Outline

- Introduction Motivations • The Calculus $\lambda_{\rightarrow \times}$ General Idea Interpretation of Types Main Point Weak Observational Preorder Application to $\lambda_{\Rightarrow\times}$
 - Application to $\lambda_{\Rightarrow\times}$

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 $\begin{array}{c} \mbox{Introduction} \\ \mbox{Reducibility Candidates} \\ \mbox{Stability by Union} \\ \mbox{Application to } \lambda_{\Rightarrow \times} \\ \mbox{Conclusion & Future work} \end{array}$

Motivations The Calculus λ_{\Rightarrow} >

Our starting point:

 Strong normalization of λ-calculus plus rewriting in presence of union types [Blanqui & Riba 06].

More generally,

- Simple characterization of reducibility candidates and saturated sets.
- Better understanding of reducibility.

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Motivations The Calculus $\lambda_{
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Motivations The Calculus $\lambda_{\Rightarrow \times}$

Terms

• Terms:

 $t, u \in \Lambda$::= $x \mid t u \mid \lambda x.t \mid \pi_i t \mid \langle t, u \rangle$.

• Reductions:

$$(\lambda \mathbf{x}.\mathbf{t})\mathbf{u}\mapsto_{\beta}\mathbf{t}[\mathbf{u}/\mathbf{x}] \qquad \pi_i \langle \mathbf{t}_1, \mathbf{t}_2 \rangle \mapsto_{\beta} \mathbf{t}_i.$$

Two kinds of values:

 $\lambda x.t$ and $\langle t, u \rangle$

Motivations The Calculus $\lambda_{\Rightarrow \times}$

Types

• Types:

$$T, U \in \mathcal{T}$$
 ::= **B** | $T \Rightarrow U$ | $T \times U$

• Typing rules:

$$(Ax) \frac{}{\Gamma, x: T \vdash x: T}$$
$$(\Rightarrow I) \frac{\Gamma, x: U \vdash t: T}{\Gamma \vdash \lambda x.t: U \Rightarrow T} \qquad (\Rightarrow E) \frac{\Gamma \vdash t: U \Rightarrow T \quad \Gamma \vdash u: U}{\Gamma \vdash tu: T}$$
$$(\times I) \frac{\Gamma \vdash t_1: T_1 \quad \Gamma \vdash t_2: T_2}{\Gamma \vdash \langle t_1, t_2 \rangle: T_1 \times T_2} \qquad (\times E) \frac{\Gamma \vdash t: T_1 \times T_2}{\Gamma \vdash \pi_i t: T_i} (i \in \{1, 2\})$$

General Idea Interpretation of Types Girard's Reducibility Candidates

Outline

Motivations • The Calculus $\lambda_{\rightarrow \times}$ **Reducibility Candidates** 2 General Idea Interpretation of Types ۲ Main Point Weak Observational Preorder Application to $\lambda_{\Rightarrow\times}$ • Application to $\lambda_{\Rightarrow\times}$

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General Idea Interpretation of Types Girard's Reducibility Candidates

- Interpret types $T \in T$ as sets of SN terms $\llbracket T \rrbracket \subseteq SN$.
- Prove the soundness of the interpretation:

If $\Gamma \vdash t : T$ and $\sigma(x) \in \llbracket A \rrbracket$ for all $(x : A) \in \Gamma$, then $\sigma(t) \in \llbracket T \rrbracket$.

• [[*T*]] must satisfy some closure conditions.

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General Idea Interpretation of Types Girard's Reducibility Candidates

• Arrow:

$$\mathcal{A} \Rightarrow \mathcal{B} =_{\mathsf{def}} \{t \mid \forall u \ (u \in \mathcal{A} \Rightarrow tu \in \mathcal{B})\}$$

Product:

$$\mathcal{A} \times \mathcal{B} =_{\mathsf{def}} \{t \mid \pi_1 t \in \mathcal{A} \land \pi_2 t \in \mathcal{B}\}$$

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General Idea Interpretation of Types Girard's Reducibility Candidates

Neutrality

Atomic elimination contexts:

 $\epsilon[] \quad ::= \quad []t \quad | \quad \pi_i[]$

- Elimination contexts: $E[] ::= [] | E[\epsilon[]].$
- *t* is *neutral* ($t \in \mathcal{N}$) iff *t* is not a value.

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If t ∈ N, then
E[t] ∈ N
If E[t] → v, then v = E'[t'] with (E[], t) → (E'[], t'
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General Idea Interpretation of Types Girard's Reducibility Candidates

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General Idea Interpretation of Types Girard's Reducibility Candidates

Definitions

• $C \in CR$ iff $C \subseteq SN$ and (CR0) if $t \in C$ and $t \to u$ then $u \in C$, (CR1) if $t \in N$ and $(\forall u (t \to u \Rightarrow u \in C))$ then $t \in C$.

• If $X \subseteq SN$, \overline{X} is the smallest set such that $X \subseteq \overline{X} \in CR$.

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General Idea Interpretation of Types Girard's Reducibility Candidates

Application

• Let $X \in \{\Rightarrow, \times\}$

If $\mathcal{A}, \mathcal{B} \in \mathcal{CR}$, then $\mathcal{A} \times \mathcal{B} \in \mathcal{CR}$.

- $\mathcal{A} \times \mathcal{B} \subseteq \mathcal{SN}$.
- $\mathcal{A} \times \mathcal{B}$ stable by reduction.

Let t ∈ N and (t) ⊆ A X B. Since ε[t] ∈ N, apply (CR1) on A, B. By induction on ε[] ∈ SN. Let (ε[], t) → v. If v = ε[t'] with t → t', we conclude by assumption. Otherwise, v = ε'[t], and we conclude by induction hypothesis.

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• Let $t \in \mathcal{N}$ and $(t)_{\rightarrow} \subseteq \mathcal{A} \times \mathcal{B}$.

Since $\epsilon[t] \in \mathcal{N}$, apply (CR1) on \mathcal{A}, \mathcal{B} . By induction on $\epsilon[] \in S\mathcal{N}$. Let $(\epsilon[], t) \to v$. If $v = \epsilon[t']$ with $t \to t'$, we conclude by assumption. Otherwise, $v = \epsilon'[t]$, and we conclude by induction hypothesis.

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General Idea Interpretation of Types Girard's Reducibility Candidates

Application

• Let
$$X \in \{\Rightarrow, \times\}$$

If $\mathcal{A}, \mathcal{B} \in C\mathcal{R}$, then $\mathcal{A} \times \mathcal{B} \in C\mathcal{R}$.

• $\mathcal{A} \times \mathcal{B} \subseteq \mathcal{SN}$.

- $\mathcal{A} \times \mathcal{B}$ stable by reduction.
- Let t ∈ N and (t) ⊆ A X B. Since ε[t] ∈ N, apply (CR1) on A, B. By induction on ε[] ∈ SN. Let (ε[], t) → V. If v = ε[t'] with t → t', we conclude by assumption. Otherwise, v = ε'[t], and we conclude by induction hypothesis.

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Main Point General Considerations Weak Observational Preorder

Outline

Motivations • The Calculus $\lambda_{\rightarrow \times}$ General Idea ۲ Interpretation of Types 3 Stability by Union Main Point Weak Observational Preorder Application to $\lambda_{\Rightarrow\times}$ • Application to $\lambda_{\Rightarrow\times}$

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Main Point General Considerations Weak Observational Preorder

• Let $C \subseteq CR$. We want

$$\bigcup \mathcal{C} =_{\mathsf{def}} \bigcup_{C \in \mathcal{C}} \mathcal{C} \in \mathcal{CR}$$

- \mathcal{SN} and $(\mathcal{CR0})$ are OK.
- (CR1)
 - Let $t \in \mathcal{N}$ with $(t)_{\rightarrow} \subseteq \bigcup \mathcal{C}$.
 - We need some $C \in C$ such that $(t)_{\rightarrow} \in C$.

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Main Point General Considerations Weak Observational Preorder

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Main Point General Considerations Weak Observational Preorder

- Let CRU be the smallest set such that $CR \subseteq CRU$ and $C \subseteq CRU \Rightarrow \bigcup C \in CRU$.
- Hence, CR is stable by union iff CR = CRU.
- Theorem 1. $C \in CRU$ iff

$$\mathsf{C} = \bigcup \{ \overline{t} \mid t \in \mathsf{C} \}$$

• Note that for all $C \in CR$,

$$C = \bigcup \{ \overline{t} \mid t \in C \}$$

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Motivations • The Calculus $\lambda_{\rightarrow \times}$ General Idea Interpretation of Types Stability by Union 3 Main Point General Considerations Weak Observational Preorder Application to $\lambda_{\Rightarrow\times}$ • Application to $\lambda_{\Rightarrow\times}$

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Main Point General Considerations Weak Observational Preorder

Definitions

• Let $t \sqsubseteq u$ iff for all value v,

$$t \to^* V \quad \Rightarrow \quad U \to^* V .$$

• Let
$$t \sqsubseteq_{SN} u$$
 iff $t \sqsubseteq u$ and $t, u \in SN$.

• We have $t \sqsubseteq u$ iff for all value v,

$\forall E[\] (E[t] \rightarrow^* v \quad \Rightarrow \quad E[u] \rightarrow^* v) .$

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Main Point General Considerations Weak Observational Preorder

Results

- Theorem 2. $\overline{t} = \{u \mid u \sqsubseteq_{SN} t\}.$
- Corollary 1. $C \in CRU$ iff

$$C = \{u \mid u \sqsubseteq_{SN} t \in C\}$$

Corollary 2. CR is stable by union iff CR is the set of all C ⊆ SN such that

$$C = \{u \mid u \sqsubseteq_{SN} t \in C\}$$

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Main Point General Considerations Weak Observational Preorder

Principal Reduct

• Therefore, CR = CRU iff for all C,

$$\mathbf{C} = \{ \boldsymbol{u} \mid \boldsymbol{u} \sqsubseteq_{\mathcal{SN}} \boldsymbol{t} \in \mathbf{C} \} \qquad \Rightarrow \qquad \mathbf{C} \in \mathcal{CR}$$

(*CR*0) Since $(u \sqsubseteq_{SN} t \land u \rightarrow u') \Rightarrow u' \sqsubseteq_{SN} t$. (*CR*1) Let $t \in N$ such that $(t)_{\rightarrow} \subseteq C$. We need some $u \in C$ such that $u \sqsubseteq_{SN} t$.

• **Theorem 3.** CR = CRU iff for every $t \in N \cap SN$, there is $u \in (t)_{\rightarrow}$ such that $t \sqsubseteq u$.

Note that $u = \max_{\sqsubseteq} (t)_{\rightarrow}$. We say that *u* is a *principal reduct* of *t*.

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Main Point General Considerations Weak Observational Preorder

Principal Reduct

• Therefore, CR = CRU iff for all C,

$$\mathbf{C} = \{ \boldsymbol{u} \mid \boldsymbol{u} \sqsubseteq_{\mathcal{SN}} \boldsymbol{t} \in \mathbf{C} \} \qquad \Rightarrow \qquad \mathbf{C} \in \mathcal{CR}$$

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Introduction Reducibility Candidates Stability by Union Application to λ⇒ × Conclusion & Future work

Application to λ_{\Rightarrow}

Outline

Motivations • The Calculus $\lambda_{\rightarrow \times}$ General Idea Interpretation of Types Main Point Weak Observational Preorder Application to $\lambda_{\Rightarrow\times}$ 4 • Application to $\lambda_{\Rightarrow\times}$

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Introduction Reducibility Candidates Stability by Union Application to λ⇒ x Conclusion & Future work

Application to $\lambda \Rightarrow \times$

In $\lambda_{\Rightarrow \times}$, we seek for principal reducts of $t \in \mathcal{N} \cap S\mathcal{N}$.

Weak Standardization:

- Let $t \rightarrow_{\beta} u$ and $E[t] \rightarrow v$ with $v \neq E[u]$.
- Then v = E'[t'] with $(E[], t) \rightarrow (E'[], t')$ and there exists u' such that $t' \rightarrow_{\beta} u'$ and $E[u] \rightarrow^* E'[u']$.

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Introduction Reducibility Candidates Stability by Union Application to λ⇒ × Conclusion & Future work

Application to $\lambda_{\Rightarrow \times}$

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Weak Standardization:

- Let $t \rightarrow_{\beta} u$ and $E[t] \rightarrow v$ with $v \neq E[u]$.
- Then v = E'[t'] with (E[], t) → (E'[], t') and there exists u' such that t' →_β u' and E[u] →^{*} E'[u'].

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In a recent paper [Riba 07]:

- We have given a characterization of the stability by union of *CR*.
- We have shown that it holds for λ⇒×, and for more elaborated calculi.
- It those cases, we have shown that Girard's Reducibility candidates are exactly the Tait's saturated sets that are stable by reduction.
- Future Work:
- Application to orthogonal rewriting.
- What happens when mixing union types and non-determinism?

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