Proof interpretations with truth

Jaime Gaspar[1,2] (joint work with Paulo Oliva[3])

[1]Technische Universität Darmstadt

[2]Supported by the Fundação para a Ciência e a Tecnologia

[3]Queen Mary, University of London

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Part I

Variants of modified realizability

Framework

Intuitionistic logic IL^{ω}

Based on $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$.

Typed We have terms of type i (integers) and b (booleans), and higher types (for example, $\mathrm{i} \rightarrow (\mathrm{b} \rightarrow \mathrm{i})$). We have constants t (true) and f (false) of type b.

λ-abstraction We have λ-abstraction: $\lambda x \,.\, t(x)$.

If-then-else We can define $A \diamondsuit_b B$ that reduces to A or B when b is t or f, respectively.

Modified realizability

Modified realizability mr,
q-variant of modified realizability mq,
modified realizability with truth mrt

$$\begin{aligned}
\text{mr } A_{\text{at}} &:\equiv A_{\text{at}} \\
\mathbf{x}, \mathbf{y} \text{ mr } (A \wedge B) &:\equiv \mathbf{x} \text{ mr } A \wedge \mathbf{y} \text{ mr } B \\
b, \mathbf{x}, \mathbf{y} \text{ mr } (A \vee B) &:\equiv \ \mathbf{x} \text{ mr } A \qquad \Diamond_b \ \ \mathbf{y} \text{ mr } B \\
\mathbf{f} \text{ mr } (A \to B) &:\equiv \forall \mathbf{x}(\mathbf{x} \text{ mr } A \qquad \to \mathbf{f}\mathbf{x} \text{ mr } B) \\
\mathbf{f} \text{ mr } \forall z A &:\equiv \forall z(\mathbf{f} z \text{ mr } A) \\
z, \mathbf{x} \text{ mr } \exists z A &:\equiv \mathbf{x} \text{ mr } A
\end{aligned}$$

Modified realizability

Modified realizability mr,
q-variant of modified realizability mq,
modified realizability with truth mrt

$$
\begin{aligned}
\text{mr } A_{\text{at}} &:\equiv A_{\text{at}} \\
\mathbf{x}, \mathbf{y} \text{ mr } (A \wedge B) &:\equiv \mathbf{x} \text{ mr } A \wedge \mathbf{y} \text{ mr } B \\
b, \mathbf{x}, \mathbf{y} \text{ mr } (A \vee B) &:\equiv (\mathbf{x} \text{ mr } A \wedge A) \Diamond_b (\mathbf{y} \text{ mr } B \wedge B) \\
\mathbf{f} \text{ mr } (A \to B) &:\equiv \forall \mathbf{x}(\mathbf{x} \text{ mr } A \wedge A \to \mathbf{f}\mathbf{x} \text{ mr } B) \\
\mathbf{f} \text{ mr } \forall z A &:\equiv \forall z(\mathbf{f} z \text{ mr } A) \\
z, \mathbf{x} \text{ mr } \exists z A &:\equiv \mathbf{x} \text{ mr } A \wedge A
\end{aligned}
$$

Modified realizability

Modified realizability mr,
q-variant of modified realizability mq,
modified realizability with truth mrt

$$
\begin{aligned}
\mathrm{mr}\ A_{\mathrm{at}} &:\equiv A_{\mathrm{at}} \\
\mathbf{x}, \mathbf{y}\ \mathrm{mr}\ (A \wedge B) &:\equiv \mathbf{x}\ \mathrm{mr}\ A \wedge \mathbf{y}\ \mathrm{mr}\ B \\
b, \mathbf{x}, \mathbf{y}\ \mathrm{mr}\ (A \vee B) &:\equiv (\mathbf{x}\ \mathrm{mr}\ A \wedge A) \lozenge_b (\mathbf{y}\ \mathrm{mr}\ B \wedge B) \\
\mathbf{f}\ \mathrm{mr}\ (A \to B) &:\equiv \forall \mathbf{x}(\mathbf{x}\ \mathrm{mr}\ A \wedge A \to \mathbf{f}\mathbf{x}\ \mathrm{mr}\ B) \wedge (A \to B) \\
\mathbf{f}\ \mathrm{mr}\ \forall z A &:\equiv \forall z(\mathbf{f} z\ \mathrm{mr}\ A) \\
z, \mathbf{x}\ \mathrm{mr}\ \exists z A &:\equiv \mathbf{x}\ \mathrm{mr}\ A \wedge A
\end{aligned}
$$

Comparison of mr, mq and mrt

Axiom of choice AC and independence of ∃-free premises IP

An ∃-free formula A_{ef} is a formula without $\vee$ and $\exists$.
AC is $\forall x \exists y A(x,y) \to \exists f \forall x A(x, f(x))$.
IP is $(A_{\mathrm{ef}} \to \exists x B) \to \exists x (A_{\mathrm{ef}} \to B)$ with x not free in A_{ef}.

Soundness theorems

If $\mathrm{IL}^{\omega} \pm \mathrm{AC} \pm \mathrm{IP} \vdash A$, then we can extract terms $\mathbf{t}$ such that

$$\mathrm{IL}^{\omega} \vdash \mathbf{t} \text{ mr } A,$$
$$\mathrm{IL}^{\omega} \pm \mathrm{AC} \pm \mathrm{IP} \vdash \mathbf{t} \text{ mq } A,$$
$$\mathrm{IL}^{\omega} \pm \mathrm{AC} \pm \mathrm{IP} \vdash \mathbf{t} \text{ mrt } A.$$

Truth properties

$\mathrm{IL}^{\omega} \vdash$	$\mathbf{x}$ mr $A \to A$	$\mathbf{x}$ mq $A \to A$	$\mathbf{x}$ mrt $A \to A$
form of A	$B_{\mathrm{ef}} \vee C_{\mathrm{ef}}$ $\exists z B_{\mathrm{ef}}$	$B \vee C$ $\exists z B$	any

Applications

Extraction of witnesses

If $\mathrm{IL}^\omega \vdash \exists z A(z)$, then we can extract a term t such that $\mathrm{IL}^\omega \vdash A(t)$:

$$\mathrm{IL}^\omega \vdash \exists z A(z)$$

$$\Downarrow \text{soundness theorem}$$

$$\mathrm{IL}^\omega \vdash t, \mathbf{q} \text{ mrt } \exists z A(z)$$

$$\Downarrow \text{definition of mrt}$$

$$\mathrm{IL}^\omega \vdash \mathbf{q} \text{ mrt } A(t)$$

$$\Downarrow \text{truth}$$

$$\mathrm{IL}^\omega \vdash A(t)$$

Applications

Extraction of choice functions

If $\mathsf{IL}^\omega \vdash \forall x \exists y A(x,y)$, then we can extract a term $t(x)$ such that $\mathsf{IL}^\omega \vdash \forall x A(x, t(x))$:

$$\mathsf{IL}^\omega \vdash \forall x \exists y A(x,y)$$

$$\Big\Downarrow \text{soundness theorem}$$

$$\mathsf{IL}^\omega \vdash t(x), \mathbf{q}(x) \text{ mrt } \forall x \exists y A(x,y)$$

$$\Big\Downarrow \text{definition of mrt}$$

$$\mathsf{IL}^\omega \vdash \forall x \big(\mathbf{q}(x) \text{ mrt } A(x, t(x))\big)$$

$$\Big\Downarrow \text{truth}$$

$$\mathsf{IL}^\omega \vdash \forall x A(x, t(x))$$

Applications

Decision of disjunctions

If $\mathsf{IL}^\omega \vdash A \vee B$, then we can decide if $\mathsf{IL}^\omega \vdash A$ or $\mathsf{IL}^\omega \vdash B$:

$$\mathsf{IL}^\omega \vdash A \vee B$$

$$\Big\Downarrow \text{soundness theorem}$$

$$\mathsf{IL}^\omega \vdash t, \mathbf{q}, \mathbf{r} \text{ mrt } (A \vee B)$$

$$\Big\Downarrow \text{definition of mrt}$$

$$\mathsf{IL}^\omega \vdash \mathbf{q} \text{ mrt } A \lozenge_t \mathbf{r} \text{ mrt } B$$

$$\Big\Downarrow \text{truth}$$

$$\mathsf{IL}^\omega \vdash A \lozenge_t B$$

$$\Big\Downarrow \text{definition of } \lozenge$$

$$\begin{cases} \mathsf{IL}^\omega \vdash A & \text{if } t = \mathrm{t} \\ \mathsf{IL}^\omega \vdash B & \text{if } t = \mathrm{f} \end{cases}$$

Framework

Intuitionistic linear logic ILL^{ω}

Based on $\otimes$, $\&$, $\oplus$, $\multimap$, $\forall$, $\exists$, $!$, $\top$, 0.

Typed As before.

λ-abstraction As before.

If-then-else As before.

Symbols and contraction

$\otimes, \&$ Are conjunctions. We have $\frac{A \vdash B \quad A \vdash C}{A, A \vdash B \otimes C}$ but $\frac{A \vdash B \quad A \vdash C}{A \vdash B \,\&\, C}$.

$\oplus$ Is a disjunction.

$\multimap$ $A \multimap B$ means that from A we get B using A exactly once (with "bookkeeping"). We have $A \not\multimap A \otimes A$ but $A \multimap A \,\&\, A$.

$!$ $!A$ means that we can use A an arbitrary number of times. So $!A \multimap B$ means that from A we get B (without "bookkeeping").

$\top, 0, !\top$ Are the identities of $\&$, $\oplus$ and $\otimes$, respectively.

Modified realizability of intuitionistic linear logic

Modified realizability of intuitionistic linear logic $|\,|$

Modified realizability with truth of linear logic $\{\,\}$

$$
\begin{aligned}
|A_{\mathrm{at}}| &:\equiv A_{\mathrm{at}} \\
|A \otimes B|^{\mathbf{x},\mathbf{v}}_{\mathbf{y},\mathbf{w}} &:\equiv |A|^{\mathbf{x}}_{\mathbf{y}} \otimes |B|^{\mathbf{v}}_{\mathbf{w}} \\
|A \,\&\, B|^{\mathbf{x},\mathbf{v}}_{\mathbf{y},\mathbf{w},b} &:\equiv |A|^{\mathbf{x}}_{\mathbf{y}} \Diamond_b |B|^{\mathbf{v}}_{\mathbf{w}} \\
|A \oplus B|^{\mathbf{x},\mathbf{v},b}_{\mathbf{y},\mathbf{w}} &:\equiv |A|^{\mathbf{x}}_{\mathbf{y}} \Diamond_b |B|^{\mathbf{v}}_{\mathbf{w}} \\
|A \multimap B|^{\mathbf{f},\mathbf{g}}_{\mathbf{x},\mathbf{w}} &:\equiv |A|^{\mathbf{x}}_{\mathbf{fxw}} \multimap |B|^{\mathbf{gx}}_{\mathbf{w}} \\
|\forall z A|^{\mathbf{f}}_{\mathbf{y},z} &:\equiv |A|^{\mathbf{f}z}_{\mathbf{y}} \\
|\exists z A|^{\mathbf{x},z}_{\mathbf{y}} &:\equiv |A|^{\mathbf{x}}_{\mathbf{y}} \\
|!A|^{\mathbf{x}} &:\equiv !\forall \mathbf{y} |A|^{\mathbf{x}}_{\mathbf{y}}
\end{aligned}
$$

Soundness theorems

If $\mathrm{ILL}^{\omega} \vdash A$, then we can extract terms $\mathbf{t}$ such that $\mathrm{ILL}^{\omega} \vdash |A|^{\mathbf{t}}_{\mathbf{y}}$ and $\mathrm{ILL}^{\omega} \vdash \{A\}^{\mathbf{t}}_{\mathbf{y}}$.

Modified realizability of intuitionistic linear logic

Modified realizability of intuitionistic linear logic $| \, |$

Modified realizability with truth of linear logic $\{ \, \}$

$$
\begin{aligned}
|A_{\mathrm{at}}| &:\equiv A_{\mathrm{at}} \\
|A \otimes B|^{\mathbf{x},\mathbf{v}}_{\mathbf{y},\mathbf{w}} &:\equiv |A|^{\mathbf{x}}_{\mathbf{y}} \otimes |B|^{\mathbf{v}}_{\mathbf{w}} \\
|A \,\&\, B|^{\mathbf{x},\mathbf{v}}_{\mathbf{y},\mathbf{w},b} &:\equiv |A|^{\mathbf{x}}_{\mathbf{y}} \Diamond_b |B|^{\mathbf{v}}_{\mathbf{w}} \\
|A \oplus B|^{\mathbf{x},\mathbf{v},b}_{\mathbf{y},\mathbf{w}} &:\equiv |A|^{\mathbf{x}}_{\mathbf{y}} \Diamond_b |B|^{\mathbf{v}}_{\mathbf{w}} \\
|A \multimap B|^{\mathbf{f},\mathbf{g}}_{\mathbf{x},\mathbf{w}} &:\equiv |A|^{\mathbf{x}}_{\mathbf{fxw}} \multimap |B|^{\mathbf{gx}}_{\mathbf{w}} \\
|\forall z A|^{\mathbf{f}}_{\mathbf{y},z} &:\equiv |A|^{\mathbf{f}z}_{\mathbf{y}} \\
|\exists z A|^{\mathbf{x},z}_{\mathbf{y}} &:\equiv |A|^{\mathbf{x}}_{\mathbf{y}} \\
|!A|^{\mathbf{x}} &:\equiv !\forall \mathbf{y} |A|^{\mathbf{x}}_{\mathbf{y}} \otimes !A
\end{aligned}
$$

Soundness theorems

If $\mathrm{ILL}^{\omega} \vdash A$, then we can extract terms $\mathbf{t}$ such that $\mathrm{ILL}^{\omega} \vdash |A|^{\mathbf{t}}_{\mathbf{y}}$ and $\mathrm{ILL}^{\omega} \vdash \{A\}^{\mathbf{t}}_{\mathbf{y}}$.

Part II

How to hardwire truth

First heuristic

Relation between mq and mrt

$\mathsf{IL}^\omega \vdash \mathbf{x} \text{ mrt } A \leftrightarrow \mathbf{x} \text{ mq } A \wedge A$.

First heuristic

If you have a q-variant of a interpretation $\mathbf{x} \text{ q } A$, attempt to find a variant with truth $\mathbf{x} \text{ t } A$ by defining

$$\mathbf{x} \text{ t } A :\equiv \mathbf{x} \text{ q } A \wedge A.$$

Second heuristic

Girard's embedding of IL^{ω} into ILL^{ω}

$$
\begin{aligned}
(A_{\mathrm{at}})^{\circ} &:\equiv\ !A_{\mathrm{at}} \\
(A \wedge B)^{\circ} &:\equiv A^{\circ} \otimes B^{\circ} \\
(A \vee B)^{\circ} &:\equiv A^{\circ} \oplus B^{\circ} \\
(A \rightarrow B)^{\circ} &:\equiv\ !(A^{\circ} \multimap B^{\circ}) \\
(\forall x A)^{\circ} &:\equiv\ !\forall x A^{\circ} \\
(\exists x A)^{\circ} &:\equiv \exists x A^{\circ}
\end{aligned}
$$

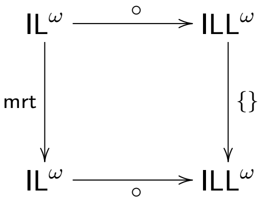
Soundness theorem

If $IL^{\omega} \vdash A$, then $ILL^{\omega} \vdash A^{\circ}$.

Second heuristic

Factorization of mrt in terms of {} and ∘

$$\mathsf{ILL}^\omega \vdash \{A^\circ\}^{\mathbf{x}} \multimapboth (\mathbf{x} \text{ mrt } A)^\circ$$



Second heuristic

Where does $\{A^\circ\}^{\mathbf{x}}$ add A's?

Since $\{A^\circ\}^{\mathbf{x}} \circ\!\!-\!\!\circ (\mathbf{x} \text{ mrt } A)^\circ$, then $\{A^\circ\}^{\mathbf{x}}$ is adding A's in the right clauses: $A_{at}, \rightarrow, \forall$.

$$
\begin{aligned}
A_{at}^\circ &:\equiv {!}A_{at} & \{A_{at}\} &:\equiv A_{at} \\
(A \wedge B)^\circ &:\equiv A^\circ \otimes B^\circ & \{A \otimes B\}^{\mathbf{x},\mathbf{v}}_{\mathbf{y},\mathbf{w}} &:\equiv \{A\}^{\mathbf{x}}_{\mathbf{y}} \otimes \{B\}^{\mathbf{v}}_{\mathbf{w}} \\
(A \vee B)^\circ &:\equiv A^\circ \oplus B^\circ & \{A \oplus B\}^{\mathbf{x},\mathbf{v},b}_{\mathbf{y},\mathbf{w}} &:\equiv \{A\}^{\mathbf{x}}_{\mathbf{y}} \Diamond_b \{B\}^{\mathbf{v}}_{\mathbf{w}} \\
(A \rightarrow B)^\circ &:\equiv {!}(A^\circ \multimap B^\circ) & \{A \multimap B\}^{\mathbf{f},\mathbf{g}}_{\mathbf{x},\mathbf{w}} &:\equiv \{A\}^{\mathbf{x}}_{\mathbf{fxw}} \multimap \{B\}^{\mathbf{gx}}_{\mathbf{w}} \\
(\forall z A)^\circ &:\equiv {!}\forall z A^\circ & \{\forall z A\}^{\mathbf{f}}_{\mathbf{y},z} &:\equiv \{A\}^{\mathbf{f}z}_{\mathbf{y}} \\
(\exists z A)^\circ &:\equiv \exists z A^\circ & \{\exists z A\}^{\mathbf{x},z}_{\mathbf{y}} &:\equiv \{A\}^{\mathbf{x}}_{\mathbf{y}} \\
& & \{!A\}^{\mathbf{x}} &:\equiv {!}\forall \mathbf{y} \{A\}^{\mathbf{x}}_{\mathbf{y}} \otimes {!}A
\end{aligned}
$$

Second heuristic

If you have a interpretation, attempt to find a variant with truth by adding A's in the clauses on $A_{at}, \rightarrow, \forall$.

Part III

Applications

Modified realizability with truth

$$\begin{aligned}
\text{mr } A_{\text{at}} &:\equiv A_{\text{at}} \\
\mathbf{x},\mathbf{y} \text{ mr } (A \wedge B) &:\equiv \mathbf{x} \text{ mr } A \wedge \mathbf{y} \text{ mr } B \\
b,\mathbf{x},\mathbf{y} \text{ mr } (A \vee B) &:\equiv \mathbf{x} \text{ mr } A \diamondsuit_b \mathbf{y} \text{ mr } B \\
\mathbf{f} \text{ mr } (A \to B) &:\equiv \forall \mathbf{x}(\mathbf{x} \text{ mr } A \to \mathbf{f}\mathbf{x} \text{ mr } B) \\
\mathbf{f} \text{ mr } \forall z A &:\equiv \forall z(\mathbf{f}z \text{ mr } A) \\
z,\mathbf{x} \text{ mr } \exists z A &:\equiv \mathbf{x} \text{ mr } A.
\end{aligned}$$

Heuristics application

1. Define $\mathbf{x} \text{ mrt } A :\equiv \mathbf{x} \text{ mq } A \wedge A$.
2. Add A's to the clauses on $A_{\text{at}}, \to, \forall$.

Modified realizability with truth

$$
\begin{aligned}
\text{mr } A_{\text{at}} &:\equiv A_{\text{at}} \wedge A_{\text{at}} \\
\mathbf{x}, \mathbf{y} \text{ mr } (A \wedge B) &:\equiv \mathbf{x} \text{ mr } A \wedge \mathbf{y} \text{ mr } B \\
b, \mathbf{x}, \mathbf{y} \text{ mr } (A \vee B) &:\equiv \mathbf{x} \text{ mr } A \Diamond_b \mathbf{y} \text{ mr } B \\
\mathbf{f} \text{ mr } (A \to B) &:\equiv \forall \mathbf{x}(\mathbf{x} \text{ mr } A \to \mathbf{f}\mathbf{x} \text{ mr } B) \wedge (A \to B) \\
\mathbf{f} \text{ mr } \forall z A &:\equiv \forall z (\mathbf{f} z \text{ mr } A) \wedge \forall z A \\
z, \mathbf{x} \text{ mr } \exists z A &:\equiv \mathbf{x} \text{ mr } A.
\end{aligned}
$$

Heuristics application

1. Define $\mathbf{x}$ mrt $A :\equiv \mathbf{x}$ mq $A \wedge A$.
2. Add A's to the clauses on $A_{\text{at}}, \to, \forall$.

Diller-Nahm interpretation with truth

$$
\begin{aligned}
(A_{\mathrm{at}})_D &:\equiv A_{\mathrm{at}} \\
(A \wedge B)_D(\mathbf{x}, \mathbf{v}; \mathbf{y}, \mathbf{w}) &:\equiv A_D(\mathbf{x}; \mathbf{y}) \wedge B_D(\mathbf{v}; \mathbf{w}) \\
(A \vee B)_D(b, \mathbf{x}, \mathbf{v}; \mathbf{y}, \mathbf{w}) &:\equiv A_D(\mathbf{x}; \mathbf{y}) \diamondsuit_b B_D(\mathbf{v}; \mathbf{w}) \\
(A \to B)_D(\mathbf{f}, \mathbf{g}, h; \mathbf{x}, \mathbf{w}) &:\equiv (\forall z < h\mathbf{x}\mathbf{w} A_D(\mathbf{x}; \mathbf{f}z\mathbf{x}\mathbf{w}) \to B_D(\mathbf{g}\mathbf{x}; \mathbf{w})) \\
(\forall z A)_D(\mathbf{f}; \mathbf{y}, z) &:\equiv A_D(\mathbf{f}\mathbf{z}; \mathbf{y}) \\
(\exists z A)_D(z, \mathbf{x}; \mathbf{y}) &:\equiv A_D(\mathbf{x}; \mathbf{y})
\end{aligned}
$$

Heuristics application

1. Define $A_{Dt}(\mathbf{x}; \mathbf{y}) :\equiv A_{Dq}(\mathbf{x}; \mathbf{y}) \wedge A$.
2. Add A's to the clauses on $A_{\mathrm{at}}, \to, \forall$.

An application: independence of universal premises rule (Jørgensen 2001)

$$\frac{\forall x A_{\mathrm{qf}} \to \exists y B}{\exists y (\forall x A_{\mathrm{qf}} \to B)}.$$

Diller-Nahm interpretation with truth

$$
\begin{aligned}
(A_{\mathrm{at}})_D &:\equiv A_{\mathrm{at}} \wedge A_{\mathrm{at}} \\
(A \wedge B)_D(\mathbf{x}, \mathbf{v}; \mathbf{y}, \mathbf{w}) &:\equiv A_D(\mathbf{x}; \mathbf{y}) \wedge B_D(\mathbf{v}; \mathbf{w}) \\
(A \vee B)_D(b, \mathbf{x}, \mathbf{v}; \mathbf{y}, \mathbf{w}) &:\equiv A_D(\mathbf{x}; \mathbf{y}) \diamondsuit_b B_D(\mathbf{v}; \mathbf{w}) \\
(A \to B)_D(\mathbf{f}, \mathbf{g}, h; \mathbf{x}, \mathbf{w}) &:\equiv (\forall z < h\mathbf{x}\mathbf{w} A_D(\mathbf{x}; \mathbf{f}z\mathbf{x}\mathbf{w}) \to B_D(\mathbf{g}\mathbf{x}; \mathbf{w})) \wedge (A \to B) \\
(\forall z A)_D(\mathbf{f}; \mathbf{y}, z) &:\equiv A_D(\mathbf{f}z; \mathbf{y}) \wedge \forall z A \\
(\exists z A)_D(z, \mathbf{x}; \mathbf{y}) &:\equiv A_D(\mathbf{x}; \mathbf{y})
\end{aligned}
$$

Heuristics application

1. Define $A_{Dt}(\mathbf{x}; \mathbf{y}) :\equiv A_{Dq}(\mathbf{x}; \mathbf{y}) \wedge A$.
2. Add A's to the clauses on $A_{\mathrm{at}}, \to, \forall$.

An application: independence of universal premises rule (Jørgensen 2001)

$$\frac{\forall x A_{\mathrm{qf}} \to \exists y B}{\exists y (\forall x A_{\mathrm{qf}} \to B)}.$$

Bounded modified realizability with truth

$$\begin{aligned}
\mathrm{br}\ A_{\mathrm{at}} &:\equiv A_{\mathrm{at}} \\
\mathbf{x},\mathbf{y}\ \mathrm{br}\ (A \wedge B) &:\equiv \mathbf{x}\ \mathrm{br}\ A \wedge \mathbf{y}\ \mathrm{br}\ B \\
\mathbf{x},\mathbf{y}\ \mathrm{br}\ (A \vee B) &:\equiv \mathbf{x}\ \mathrm{br}\ A \vee \mathbf{y}\ \mathrm{br}\ B \\
\mathbf{f}\ \mathrm{br}\ (A \to B) &:\equiv \tilde{\forall}\mathbf{x}(\mathbf{x}\ \mathrm{br}\ A \to \mathbf{fx}\ \mathrm{br}\ B) \\
\mathbf{x}\ \mathrm{br}\ \exists z \leq^* tA &:\equiv \exists z \leq^* t(\mathbf{x}\ \mathrm{br}\ A) \\
\mathbf{x}\ \mathrm{br}\ \forall z \leq^* tA &:\equiv \forall z \leq^* t(\mathbf{x}\ \mathrm{br}\ A) \\
\mathbf{f}\ \mathrm{br}\ \forall zA &:\equiv \tilde{\forall}u\forall z \leq^* u(\mathbf{f}u\ \mathrm{br}\ A) \\
u,\mathbf{x}\ \mathrm{br}\ \exists zA &:\equiv \exists z \leq^* u(\mathbf{x}\ \mathrm{br}\ A)
\end{aligned}$$

Heuristics application

1. Define $\mathbf{x}\ \mathrm{brt}\ A :\equiv \mathbf{x}\ \mathrm{brq}\ A \wedge A$.
2. Add A's to the clauses on $A_{\mathrm{at}}, \to, \forall$.

An application: bounded independence of $\tilde{\exists}$-free premises rule

If $A_{\tilde{\mathrm{e}}\mathrm{f}} \to \exists xB$ is a sentence, then $\dfrac{A_{\tilde{\mathrm{e}}\mathrm{f}} \to \exists xB}{\tilde{\exists}y(A_{\tilde{\mathrm{e}}\mathrm{f}} \to \exists x \leq^* yB)}$.

Bounded modified realizability with truth

$$
\begin{aligned}
\text{br } A_{\text{at}} &:\equiv A_{\text{at}} \wedge A_{\text{at}} \\
\mathbf{x}, \mathbf{y} \text{ br } (A \wedge B) &:\equiv \mathbf{x} \text{ br } A \wedge \mathbf{y} \text{ br } B \\
\mathbf{x}, \mathbf{y} \text{ br } (A \vee B) &:\equiv \mathbf{x} \text{ br } A \vee \mathbf{y} \text{ br } B \\
\mathbf{f} \text{ br } (A \to B) &:\equiv \tilde{\forall} \mathbf{x}(\mathbf{x} \text{ br } A \to \mathbf{fx} \text{ br } B) \wedge (A \to B) \\
\mathbf{x} \text{ br } \exists z \leq^* tA &:\equiv \exists z \leq^* t(\mathbf{x} \text{ br } A) \\
\mathbf{x} \text{ br } \forall z \leq^* tA &:\equiv \forall z \leq^* t(\mathbf{x} \text{ br } A) \\
\mathbf{f} \text{ br } \forall z A &:\equiv \tilde{\forall} u \forall z \leq^* u(\mathbf{f} u \text{ br } A) \wedge \forall z A \\
u, \mathbf{x} \text{ br } \exists z A &:\equiv \exists z \leq^* u(\mathbf{x} \text{ br } A)
\end{aligned}
$$

Heuristics application

1. Define $\mathbf{x}$ brt $A :\equiv \mathbf{x}$ brq $A \wedge A$.
2. Add A's to the clauses on $A_{\text{at}}, \to, \forall$.

An application: bounded independence of $\tilde{\exists}$-free premises rule

If $A_{\tilde{\exists}\text{f}} \to \exists x B$ is a sentence, then $\dfrac{A_{\tilde{\exists}\text{f}} \to \exists x B}{\tilde{\exists} y (A_{\tilde{\exists}\text{f}} \to \exists x \leq^* y B)}$.

Bounded functional interpretation with truth

$$
\begin{aligned}
(A_{\text{at}})_B &:\equiv A_{\text{at}} \\
(A \wedge B)_B(\mathbf{x},\mathbf{v};\mathbf{y},\mathbf{w}) &:\equiv A_B(\mathbf{x};\mathbf{y}) \wedge B_B(\mathbf{v};\mathbf{w}) \\
(A \vee B)_B(\mathbf{x},\mathbf{v};\mathbf{y},\mathbf{w}) &:\equiv \tilde{\forall}\tilde{\mathbf{y}} \trianglelefteq \mathbf{y} A_B(\mathbf{x};\tilde{\mathbf{y}}) \vee \tilde{\forall}\tilde{\mathbf{w}} \trianglelefteq \mathbf{w} B_B(\mathbf{v};\tilde{\mathbf{w}}) \\
(A \to B)_B(\mathbf{f},\mathbf{g};\mathbf{x},\mathbf{w}) &:\equiv (\tilde{\forall}\mathbf{y} \trianglelefteq \mathbf{fxw} A_B(\mathbf{x};\mathbf{y}) \to B_B(\mathbf{gx};\mathbf{w})) \\
(\forall z \trianglelefteq t A)_B(\mathbf{x};\mathbf{y}) &:\equiv \forall z \trianglelefteq t A_B(\mathbf{x};\mathbf{y}) \\
(\exists z \trianglelefteq t A)_B(\mathbf{x};\mathbf{y}) &:\equiv \exists z \trianglelefteq t \tilde{\forall}\tilde{\mathbf{y}} \trianglelefteq \mathbf{y} A_B(\mathbf{x};\tilde{\mathbf{y}}) \\
(\forall z A)_B(\mathbf{f};u,\mathbf{y}) &:\equiv \forall z \trianglelefteq u A_B(\mathbf{f}u;\mathbf{y}) \\
(\exists z A)_B(u,\mathbf{x};\mathbf{y}) &:\equiv \exists z \trianglelefteq u \tilde{\forall}\tilde{\mathbf{y}} \trianglelefteq \mathbf{y} A_B(\mathbf{x};\tilde{\mathbf{y}})
\end{aligned}
$$

Heuristics application

1. Define $A_{Bt}(\mathbf{x};\mathbf{y}) :\equiv A_{Bq}(\mathbf{x};\mathbf{y}) \wedge A$.
2. Add A's to the clauses on $A_{\text{at}}, \to, \forall$.

An application: bounded rule of choice

If $\forall x \exists y A$ is a sentence, then $\dfrac{\forall x \exists y A}{\tilde{\exists} V \tilde{\forall} u \forall x \trianglelefteq u \exists y \trianglelefteq V u A}$.

Bounded functional interpretation with truth

$$
\begin{aligned}
(A_{\mathrm{at}})_B &:\equiv A_{\mathrm{at}} \wedge A_{\mathrm{at}} \\
(A \wedge B)_B(\mathbf{x},\mathbf{v};\mathbf{y},\mathbf{w}) &:\equiv A_B(\mathbf{x};\mathbf{y}) \wedge B_B(\mathbf{v};\mathbf{w}) \\
(A \vee B)_B(\mathbf{x},\mathbf{v};\mathbf{y},\mathbf{w}) &:\equiv \tilde{\forall}\tilde{\mathbf{y}} \trianglelefteq \mathbf{y} A_B(\mathbf{x};\tilde{\mathbf{y}}) \vee \tilde{\forall}\tilde{\mathbf{w}} \trianglelefteq \mathbf{w} B_B(\mathbf{v};\tilde{\mathbf{w}}) \\
(A \to B)_B(\mathbf{f},\mathbf{g};\mathbf{x},\mathbf{w}) &:\equiv (\tilde{\forall}\mathbf{y} \trianglelefteq \mathbf{fxw} A_B(\mathbf{x};\mathbf{y}) \to B_B(\mathbf{gx};\mathbf{w})) \wedge (A \to B) \\
(\forall z \trianglelefteq tA)_B(\mathbf{x};\mathbf{y}) &:\equiv \forall z \trianglelefteq t A_B(\mathbf{x};\mathbf{y}) \\
(\exists z \trianglelefteq tA)_B(\mathbf{x};\mathbf{y}) &:\equiv \exists z \trianglelefteq t \tilde{\forall}\tilde{\mathbf{y}} \trianglelefteq \mathbf{y} A_B(\mathbf{x};\tilde{\mathbf{y}}) \\
(\forall z A)_B(\mathbf{f};u,\mathbf{y}) &:\equiv \forall z \trianglelefteq u A_B(\mathbf{f}u;\mathbf{y}) \wedge \forall z A \\
(\exists z A)_B(u,\mathbf{x};\mathbf{y}) &:\equiv \exists z \trianglelefteq u \tilde{\forall}\tilde{\mathbf{y}} \trianglelefteq \mathbf{y} A_B(\mathbf{x};\tilde{\mathbf{y}})
\end{aligned}
$$

Heuristics application

1. Define $A_{Bt}(\mathbf{x};\mathbf{y}) :\equiv A_{Bq}(\mathbf{x};\mathbf{y}) \wedge A$.
2. Add A's to the clauses on $A_{\mathrm{at}}, \to, \forall$.

An application: bounded rule of choice

If $\forall x \exists y A$ is a sentence, then $\dfrac{\forall x \exists y A}{\tilde{\exists} V \tilde{\forall} u \forall x \trianglelefteq u \exists y \trianglelefteq VuA}$.

Summary

- mr has two truth variants: mq and mrt.
- Case studying them we got two heuristics:
  - $\mathbf{x}$ mrt $A \leftrightarrow \mathbf{x}$ mq $A \wedge A \;\rightsquigarrow\; \mathbf{x}$ t $A :\equiv \mathbf{x}$ q $A \wedge A$;
  - $\{A^\circ\}^{\mathbf{x}} \multimapboth (\mathbf{x} \text{ mrt } A)^\circ \;\rightsquigarrow\;$ add A's to $A_{\mathrm{at}}, \rightarrow, \forall$.
- The heuristics work on:
  - modified realizability;
  - Diller-Nahm functional interpretation;
  - bounded modified realizability;
  - bounded functional interpretation.