Realizability toposes and the tripos-to-topos construction

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Kleene's number realizability

Goal: extract algorithmic information from proofs in Heyting arithmetic

- Language of Heyting arithmetic: single-sorted first order predicate logic with constant symbols for all *primitive recursive* functions
- Logic for Heyting arithmetic: intuitionistic logic with induction on arbitrary formulas

For a formal treatment, see e.g. [TvD88]

[TvD88] A. S. Troelstra and D. van Dalen, *Constructivism in mathematics. Vol. I*, Studies in Logic and the Foundations of Mathematics, vol. 121, North-Holland Publishing Co., Amsterdam, 1988.

Kleene's number realizability

- To each closed formula of φ of Heyting arithmetic, we want to associate a set $\mathrm{rn}(\varphi) \subseteq \mathbb{N}$ of *realizers*, to be viewed as codes (Gödelnumbers) of algorithms
- Notations:
  - $(n, m) \mapsto \langle n, m \rangle$ is a primitive recursive coding of pairs, with corresponding projections p_0 and p_1 (i.e., $p_0(\langle n, m \rangle) = n$ and $p_1(\langle n, m \rangle) = m$)
  - $\{n\}(m)$ ('Kleene brackets') denotes the (only partially defined) evaluation of the nth partial recursive function at input m. For this to make sense, we assume some recursive enumeration $\varphi_n(\cdot)$ of partial recursive functions.
  - When claiming $s = t$ where s, t are terms possibly containing Kleene brackets, we assert in particular that both terms are defined.

Kleene's number realizability

Now we can define $\mathrm{rn}(\varphi)$ by induction over the structure of φ.

$n \Vdash s = t$	iff	$s = t$ (s, t are closed terms)
$n \Vdash \varphi \wedge \psi$	iff	$p_0(n) \Vdash \varphi$ and $p_1(n) \Vdash \psi$
$n \Vdash \varphi \Rightarrow \psi$	iff	$\forall m \,.\, (m \Vdash \varphi) \Rightarrow (\{n\}(m) \Vdash \psi)$
$n \Vdash \bot$	never	
$n \Vdash \varphi \vee \psi$	iff	$(p_0(n) = 0 \wedge p_1(n) \Vdash \varphi) \vee (p_0(n) = 1 \wedge p_1(n) \Vdash \psi)$
$n \Vdash \forall x \,.\, \varphi(x)$	iff	$\forall m \in \mathbb{N} \,.\, \{n\}(m) \Vdash \varphi(\underline{m})$
$n \Vdash \exists x \,.\, \varphi(x)$	iff	$p_1(n) \Vdash \varphi(\underline{p_0(n)})$

(For $n \in \mathbb{N}$, $\underline{n}$ denotes the term $\underbrace{S(\dots S}_{n \text{ times}}(0)\dots)$ of Heyting arithmetic)

The effective tripos

- Now we will reformulate the ideas of Kleene realizability in a categorical setting, which will lead us to the concept of *tripos*.
- A tripos is a certain kind of *fibration*. Fibrations are of interest in *categorical logic* because they allow to model logics and type theories. In this setting, triposes correspond to *intuitionistic higher order logic*.

The effective tripos

Truth values, predicates

- Truth values in number realizability are sets of natural numbers
- Given a set I, a *predicate on I* is a function $\varphi : I \to \mathcal{P}(\mathbb{N})$. We denote the set of predicates on I by $\mathbf{eff}(I)$.
- On $\mathbf{eff}(I)$ we can define define a preorder $\vdash_I$ by

$$\varphi \vdash_I \psi \quad \text{iff} \quad \exists e \in \mathbb{N}\, \forall i \in I\, \forall n \in \varphi(i)\,.\, \{e\}(n) \in \psi(i)$$

The effective tripos

Reindexing / substitution

- For each $u : I \to J$, we can define a function

  $$\mathbf{eff}(u) : \mathbf{eff}(J) \to \mathbf{eff}(I), \qquad \varphi \mapsto \varphi \circ u$$

  $\mathbf{eff}(u)$ is monotonic with respect to $\vdash_J, \vdash_I$.
- The assignment $u \mapsto \mathbf{eff}(u)$ is furthermore compatible with composition and maps identities to identities, and thus we obtain a *contravariant functor*

  $$\mathbf{eff} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Preord}$$

  This functor is called *the effective tripos*. (A formal definition of tripos will come later)

The effective tripos

Propositional connectives

- We will now describe how to interpret predicate logic in the effective tripos. We begin with the propositional part.
- We define operations on truth values (subsets of $\mathbb{N}$) corresponding to propositional connectives.
  $\wedge, \vee, \Rightarrow : \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ and $\bot \in \mathcal{P}\mathbb{N}$ are given by

$$
\begin{aligned}
M \wedge N &= \{n \mid p_0(n) \in M, p_1(n) \in N\} \\
M \vee N &= \{n \mid p_0(n) = 0 \wedge p_1(n) \in M \vee p_0(n) = 1 \wedge p_1(n) \in N\} \\
M \Rightarrow N &= \{e \mid \forall n \in M \,.\, \{e\}(n) \in N\} \\
\bot &= \varnothing
\end{aligned}
$$

The effective tripos

Propositional connectives

- We can extend the definitions of the previous slide from truth vaules to predicates by applying them pointwise, i.e., $(\varphi \wedge \psi)(i) := \varphi(i) \wedge \psi(i)$ and so forth.
- This makes $\mathbf{eff}(I)$ into a pre- Heyting algebra, that is a distributive pre-lattice with a binary operation $\Rightarrow$ satisfying

$$\varphi \wedge \psi \vdash_I \gamma \quad \text{iff} \quad \varphi \vdash_I \psi \Rightarrow \gamma$$

- We remark that we really have no choice in defining the propositional connectives.
  They are uniquely determined (up to $\simeq$) by universal properties!

The effective tripos

Equality

On each set I we define the following equality predicate $\mathrm{eq}_I \in \mathbf{eff}(I \times I)$.

$$\mathrm{eq}_I(i,j) = \begin{cases} \mathbb{N} & i = j \\ \varnothing & \text{else} \end{cases}$$

- Quantification should correspond on the semantic level to operations of type $\forall, \exists : \mathbf{eff}(I \times J) \to \mathbf{eff}(I)$, subject to the relations

$$\begin{array}{lll} \varphi \vdash_I \forall(\psi) & \text{iff} & \varphi \circ \pi \vdash_{I \times J} \psi, \qquad \text{and} \\ \exists(\psi) \vdash_I \varphi & \text{iff} & \psi \vdash_{I \times J} \varphi \circ \pi, \end{array}$$

  where $\varphi \in \mathbf{eff}(I)$, $\psi \in \mathbf{eff}(I \times J)$, and $\pi : I \times J \to I$ is the first projection.
- We consider quantification not only along projections, but along arbitrary morphisms $u : J \to I$. The governing relations are then

$$\begin{array}{lll} \varphi \vdash_I \forall_u(\psi) & \text{iff} & \varphi \circ u \vdash_J \psi, \qquad \text{and} \\ \exists_u(\psi) \vdash_I \varphi & \text{iff} & \psi \vdash_J \varphi \circ u. \end{array}$$

The effective tripos

Quantification

Quantification in the effective tripos is given as follows.

$$(\forall_u \varphi)(i) = \bigcap_{j \in J} \mathrm{eq}(uj, i) \Rightarrow \varphi(j)$$

$$(\exists_u \varphi)(i) = \bigcup_{j \in J} \mathrm{eq}(uj, i) \wedge \varphi(j)$$

Here $u : J \to I$ and $\varphi \in \mathbf{eff}(J)$.

The effective tripos

Interpreting predicate logic

- Consider a language of many sorted predicate logic with sort symbols $S_1, \ldots, S_n$, function symbols $f_i, i \in I$ of specified arities and relation symbols $R_j, j \in J$ of specified arities.
- Assign sets $[\![S]\!]$ to sorts S, functions $[\![f]\!] : [\![S_1]\!] \times \cdots \times [\![S_n]\!] \to [\![S]\!]$ to to each function symbol f of arity $S_1 \times \cdots \times S_n \to S$, and predicates $[\![R]\!] \in \mathbf{eff}([\![S_1]\!] \times \cdots \times [\![S_m]\!]$ to each relation symbol R of arity $S_1 \times \cdots \times S_m$.
- Now we can assign by structural induction
  - to each term $x_1{:}S_1, \ldots, x_n{:}S_n \mid t : S$ in context a function $[\![t]\!] : [\![S_1]\!] \times \cdots \times [\![S_n]\!] \to [\![S]\!]$,
  - and to each formula $x_1{:}S_1, \ldots, x_n{:}S_n \mid \varphi$ in context a predicate $[\![\varphi]\!] \in \mathbf{eff}([\![S_1]\!] \times \cdots \times [\![S_n]\!])$.

The effective tripos

Soundness of the interpretation

If a formula $x_1{:}S_1, \ldots, x_n{:}S_n \mid \varphi$ is derivable in intuitionistic predicate logic,

$$[\![\varphi]\!] \simeq \top \quad \text{in} \quad \mathbf{eff}([\![S_1]\!] \times \cdots \times [\![S_n]\!])$$

The effective tripos

The internal language

- The internal language is the language which has a sort symbols for all sets, function symbols for all functions, and relation symbols for all predicates of **eff**.
- The internal language is the appropriate tool to do calculations in the effective tripos.

Triposes

More generally, the previously described way to interpret predicate logic works in arbitrary *triposes*.

So what is a tripos?

Definition of Tripos

Let $\mathcal{C}$ be a cartesian closed category. A *tripos over* $\mathcal{C}$ is a (pseudo-)functor

$$\mathcal{P} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Preord}$$

such that

1. All $\mathcal{P}(C)$ are pre-Heyting algebras
2. For $f : I \to J$ in $\mathcal{C}$, the monotone mapping $\mathcal{P}(f) : \mathcal{P}(J) \to \mathcal{P}(I)$ preserves all structure of pre-Heyting algebras
3. For all $f : A \to B$ in $\mathcal{C}$, the reindexing map $\mathcal{P}(f) : \mathcal{P}_B \to \mathcal{P}_A$ has left and right adjoints

   $$\exists_f \dashv f^* \dashv \forall_f$$

   satisfying the Beck-Chevalley condition.
4. $\mathcal{P}$ has a *generic predicate*, that is a predicate $\mathrm{tr} \in \mathcal{P}(\mathrm{Prop})$ such that for all $I \in \mathrm{Obj}(\mathcal{C})$ and all $\varphi \in \mathcal{P}(I)$ there exists a (not necessarily unique) morphism $\ulcorner \varphi \urcorner : I \to \mathrm{Prop}$ such that $\mathcal{P}(\ulcorner \varphi \urcorner)(\mathrm{tr}) \simeq \varphi$.

We want to describe now how to obtain a topos from a tripos. For this, we first of all given a definition of topos.

Definition of topos

A topos is a category with finite limits, exponentials (a.k.a. internal homs) and a subobject classifier.

The tripos-to-topos construction

The topos $\boldsymbol{T}\mathcal{P}$

For a tripos $\mathcal{P}$ on $\mathcal{C}$, we can construct a topos $\boldsymbol{T}\mathcal{P}$ as follows:

The **objects** of $\boldsymbol{T}\mathcal{P}$ are pairs $A = (|A|, \sim_A)$, where $|A| \in \mathrm{Obj}(\mathcal{C})$, $(\sim_A) \ \in \mathcal{P}(|A| \times |A|)$, and the judgements

$$x \sim_A y \vdash y \sim_A x$$
$$x \sim_A y, y \sim_A z \vdash x \sim_A z$$

hold in the logic of $\mathcal{P}$

Intuition: "$\sim_A$ is a partial equivalence relation on $|A|$ in the logic of $\mathcal{P}$"

The tripos-to-topos construction

The topos $\boldsymbol{T}\mathcal{P}$ (continued)

Morphisms of $\boldsymbol{T}\mathcal{P}$ are given by functional relations with respect to $\mathcal{P}$.

More precisely, a morphism from A to B is a $(\dashv\vdash)$-equivalence class of predicates on $|A| \times |B|$ such that for some (or equivalently any) representative ϕ the following judgements hold in $\mathcal{P}$.

$$
\begin{aligned}
\phi(x, y) &\vdash x \sim_A x \wedge y \sim_B y \\
\phi(x, y), x \sim_A x', y \sim_B y' &\vdash \phi(x', y') \\
\phi(x, y), \phi(x, y') &\vdash y \sim_B y' \\
x \sim_A x &\vdash \exists y \,.\, \phi(x, y)
\end{aligned}
$$

The tripos-to-topos construction

The topos $\boldsymbol{T}\mathcal{P}$ (continued)

Given morphisms

$$A \xrightarrow{[\phi]} B \xrightarrow{[\gamma]} C\,,$$

their **composition** is given by $[\gamma \circ \phi]$, where $\gamma \circ \phi \in \mathcal{P}_{|A| \times |C|}$ is the predicate

$$x, z \mid \exists y\,.\,\phi(x, y) \wedge \gamma(y, z).$$

The **identity** morphism on A is $[\sim_A]$.

The tripos-to-topos construction (comment)

- The construction only uses regular logic (conjunction and existential quantification), however we need full higher order logic to obtain a topos.

The effective topos

- The topos that we obtain when we apply the tripos-to-topos construction to the effective tripos is called the effective topos, denoted by $\mathcal{E}\!f\!f$.
- $\mathcal{E}\!f\!f$ can be viewed as 'the universe of recursive mathematics'
- $\mathcal{E}\!f\!f$ has a natural numbers object, which we denote by $\mathbf{N}$.
  The morphisms $f : \mathbf{N} \rightarrow \mathbf{N}$ are precisely the total recursive functions.
  More generally, morphisms between objects generated from $\mathbf{N}$ by products and arrow types (i.e. the *finite type hierarchy*) correspond precisely to the hereditarily effective operations.
- $\mathcal{E}\!f\!f$ gives rise to nice models of System F and the Calculus of Constructions.

But can we do all this also with λ-terms?

Yes, but we obtain a *different* topos.
This means that equivalent computability models can give rise to different toposes, in other words there are different universes of recursive mathematics.

We now want to describe how the previously described construction can be characterized by a universal property.

It will turn out that the tripos-to-topos construction is in a certain 2-dimensional sense left adjoint to a forgetful functor from toposes to triposes

To make this precise, we have to define the 2-categories that we want to work in.

2-categories of toposes

What should be the one-cells?

Possible choices:

- Logical functors : Too restrictive
- Geometric morphisms : Good, but the tentative unit of the biadjunction we want to present is not a geometric morphism
- Cartesian (finite limit preserving) functors : Right choice
- Regular functors : Have special status among cartesian functors

Geometric morphisms can be recovered later as adjunctions of cartesian functors.

Tripos morphisms

From now on, we will view triposes as fibrations instead of presheafs, by means of the *Grothendieck construction*.

Given triposes $\mathcal{P} : \mathcal{X} \to \mathcal{C}$ and $\mathcal{Q} : \mathcal{Y} \to \mathcal{D}$, a morphism between them is a pair

$$(F : \mathcal{C} \to \mathcal{D}, \quad \Phi : \mathcal{X} \to \mathcal{Y})$$

of functors such that

1. The diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{Y} \\ {\scriptstyle \mathcal{P}}\downarrow & & \downarrow{\scriptstyle \mathcal{Q}} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array}$$

   commutes (on the nose).
2. Φ maps cartesian arrows to cartesian arrows.
3. F preserves finite limits and Φ preserves finite meets.

If Φ furthermore commutes with existential quantification, then we call the tripos morphism *regular*.

2-cells of triposes

A 2-cell

$$\eta : (F, \Phi) \to (G, \Gamma) : \mathcal{P} \to \mathcal{Q}$$

is a natural transformation

$$\eta : F \to G$$

such that for all $A \in \mathrm{Obj}(\mathcal{C})$ and all $\psi \in \mathrm{Obj}(\mathcal{P}_A)$, we have

$$x \mid (\Phi\psi)(x) \vdash (\Gamma\psi)(\eta_A(x))$$

in the logic of $\mathcal{Q}$.

Embedding toposes into triposes

For a given category $\mathcal{C}$, we denote by $\boldsymbol{M}(\mathcal{C})$ the full subcategory of $\mathcal{C}{\downarrow}\mathcal{C}$ on the monomorphisms.

For each topos $\mathcal{E}$, its *subobject fibration*

$$\partial_1{}^{1} : \boldsymbol{M}(\mathcal{E}) \to \mathcal{E}$$

is a tripos, which we denote by $\boldsymbol{S}\mathcal{E}$.

It is straightforward to check that this gives rise to a 2-functor $\boldsymbol{S}$ from toposes to triposes.

[1] ∂_1 is the codomain projection

The tripos-to-topos construction

Mapping tripos morphisms to functors between toposes

Now that we know what a tripos morphism is, we can try to define how the tripos-to-topos construction maps a tripos morhpism to a functor between toposes.

- Easy for *regular* tripos morphisms:
  Given a regular tripos morphism

  $$(F, \Phi) : \mathcal{P} \to \mathcal{Q},$$

  the functor

  $$\boldsymbol{T}(F, \Phi) : \boldsymbol{T}\mathcal{P} \to \boldsymbol{T}\mathcal{Q}$$

  is given by

  $$\begin{array}{rcc} (|A|, \sim_A) & \mapsto & (F(|A|), \Phi(\sim_A)) \\ ([\phi] : (|A|, \sim_A) \to (|B|, \sim_B)) & \mapsto & [\Phi\phi] \end{array}$$

The tripos-to-topos construction

Mapping tripos morphisms to functors between toposes

- This method does not work if (F, Φ) is not regular, because then, $\Phi\phi$ is not total in general
- Interestingly, this can be circumvented by using a completion process for objects in $\boldsymbol{T}\mathcal{P}$.
- Construction becomes more clumsy
- Find an elegant *characterization*!

Motivating example

- Every complete Heyting algebra A give rise to a tripos $\tilde{A}$ over **Set**:
  - Fibre over I is A^I
  - Reindexing is given by precomposition
- Meet preserving maps between complete Heyting algebras give rise to tripos morphisms

Consider the succession of tripos morphisms

$$\tilde{\mathbb{B}} \xrightarrow{\tilde{\delta}} \widetilde{\mathbb{B} \times \mathbb{B}} \xrightarrow{\tilde{\wedge}} \tilde{\mathbb{B}},$$

where $\mathbb{B} = \{\text{true}, \text{false}\}$ with false $\leq$ true.
What do we get when applying the tripos-to-topos construction?

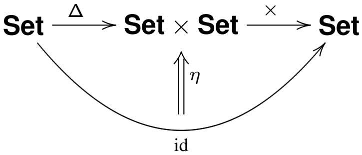
Motivating example

Answer:

$$\mathbf{Set} \xrightarrow{\Delta} \mathbf{Set} \times \mathbf{Set} \xrightarrow{\times} \mathbf{Set}\,,$$

Motivating example

However, the composition gets mapped to the identity functor!



The tripos-to-topos construction seems to be an *oplax* functor!

Towards a universal characterization of the tripos-to-topos construction

- We want to characterize the tripos-to-topos construction as being left adjoint to $\boldsymbol{S}$ (the forgetful functor from toposes to triposes)
- This can not be an ordinary biadjunction, as the tripos-to-topos construction seems to be oplax, and ordinary biadjunctions live in the framework of bicategories and *pseudofunctors*.
- However, we still have something that looks like a unit and gives rise to a 'universal lifting property' (explained below).

Towards a universal characterization of the tripos-to-topos construction

The ‘unit’ of the ‘adjunction’

For each tripos $\mathcal{P} : \mathcal{X} \to \mathcal{C}$, there is a tripos transformation

$$(D, \Xi) : \mathcal{P} \to \boldsymbol{ST}\mathcal{P}$$

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ \Xi\ } & \boldsymbol{M}(\boldsymbol{T}\mathcal{P}) \\ {\scriptstyle \mathcal{P}}\downarrow & & \downarrow{\scriptstyle \partial_1} \\ \mathcal{C} & \xrightarrow[\ D\ ]{} & \boldsymbol{T}\mathcal{P} \end{array}$$

D is the so-called ‘constant objects functor’, it is defined as

$$\begin{aligned} A &\mapsto (A, =_A) \\ f &\mapsto [x, y \mid f(x) = y] \end{aligned}$$

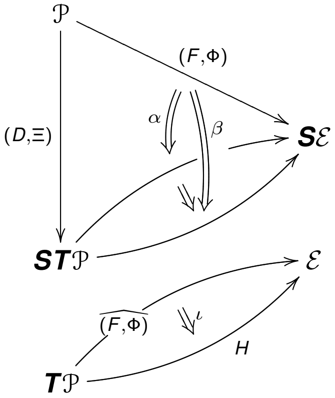
Exercise: For the definition of Ξ, make yourself clear how one can associate subobjects of DA to predicates on A in $\mathcal{P}$.

The universal lifting property

It turns out that that we have a lifting property for (D, Ξ) that has a slight resemblance to the condition for left adjointability of functors in one dimension.

For each tripos morphism (F, Φ), there is a cartesian functor $\widehat{(F, \Phi)}$ and a tripos transformation α such that for all H and β, there is a unique mediating ι.

In other words, the category $(\mathcal{P} \swarrow \boldsymbol{S})((D, \Xi), (F, \Phi))$ has an initial object $(\widehat{(F, \Phi)}, \alpha)$.



The universal lifting property

The universal lifting property suffices to construct an oplax functor, however it does *not* determine the tentative unit (D, Ξ) up to equivalence.

We will now define a three-dimensional category in which the 2-category of triposes and the 2-category of triposes are objects, and the tripos-to-topos construction is an ordinary biadjunction.

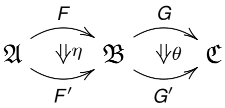
In this structure, the above ‘universal lifting property’ will be part of a characterization of *left adjointablility*.

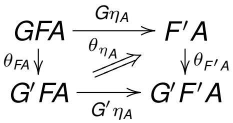
In comparison to the tripos-to-topos construction, we will from now on revert all 2-cells, such that everything is *lax* instead of *oplax*

dc-categories

- The canonical tricategory is given by bicategories, pseudofunctors, pseudo-natural transformations and modifications.
- When we try to define a tricategory out of lax functors and lax transformations, we run into two problems:

First problem: Given pseudofunctors F, F', G, G' and lax transformations η, θ as in the left diagram below, there are two generally non-isomorphic ways to define $(\theta \circ \eta)_A : GFA \to G'F'A$:





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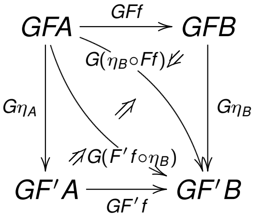
dc-categories

Second problem: If the functor G is also lax, then the composition $G \circ \eta$ is not even definable! If we try to compose constraint cells of G and η to construct the constraint cell $(G\eta)_f$, we run into a problem:



dc-categories avoid these problems while still having lax features!

dc-categories

Definition

A **dc-category** is just a 2-category $\mathfrak{A}$ together with a designated subclass $\mathfrak{A}_r$ of the class of all 1-cells such that

- $\mathfrak{A}_r$ contains all equivalences,
- $\mathfrak{A}_r$ is closed under composition, and
- $\mathfrak{A}_r$ is closed under vertical isomorphisms; i.e if $f \in \mathfrak{A}_r$ and $f \cong g$, then $g \in \mathfrak{A}_r$.

We call the arrows in $\mathfrak{A}_r$ *regular* arrows, and denote them by '$\longmapsto$' in diagrams.

A **semi-lax functor** between dc-categories $\mathfrak{A}$ and $\mathfrak{B}$ is a lax functor $(F, \phi) : \mathfrak{A} \to \mathfrak{B}$ such that

- F maps regular arrows in $\mathfrak{A}$ to regular arrows in $\mathfrak{B}$,
- all $\phi_A : \mathrm{id}_{FA} \to F(\mathrm{id}_A)$ are invertible,
- $\phi_{(f,g)} : Fg \circ Ff \to F(g \circ f)$ is invertible whenever g is regular

A **semi-lax transformation** between semi-lax functors F, G is a lax natural transformation $\eta : F \to G$ such that

- For each object A, η_A is regular, and
- η_f is invertible whenever f is regular.

dc-categories

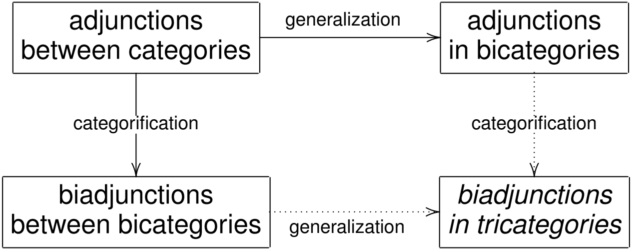
Exercise: Verify that semi-lax functors and transformations can be composed just like pseudofunctors and pseudo-natural transformations. In particular, check that the disturbing 2-cells mentioned 3 slides earlier become invertible.

Conjecture: dc-categories, semi-lax functors, semi-lax transformations and modifications form a *tricategory*.

This seems reasonable, because semi-lax functors and transformations are very similar to pseudofunctors and pseudo-natural transformations in their behaviour. (However, I did not even manage to comprehend the proof that the pseudofunctors and pseudo-natural transformations form a tricategory)

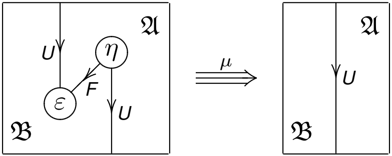
Abstract biadjunctions

- Adjunctions between categories can be generalized to adjunctions in bicategories, and they can be categorified to adjunctions between bicategories.
- If we combine these processes, we get *biadjunctions in tricategories*.



Abstract biadjunctions

To categorify the definition of adjunctions via triangle-equalities, we replace the triangle equalities by isomorphic *3-cells*



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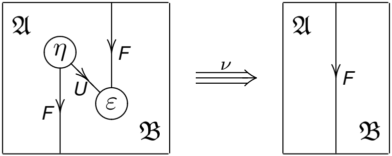
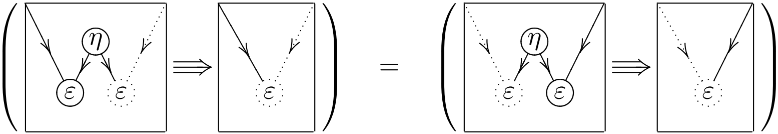


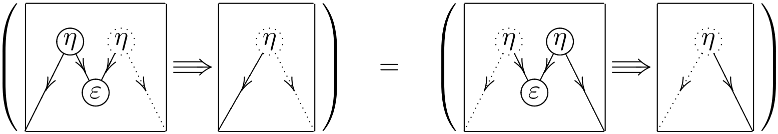
The most interesting question is ‘What are the new axioms?’

Abstract biadjunctions

In semi string diagram style, the axioms for abstract biadjunctions are



and



This elegant and comprehensible representation is due to John Baez [HDA4], if we write the equations out as pasting diagrams or even purely symbolic, things get badly readable because of the constraint cells.

Semi-lax adjunctions

A *semi-lax adjunction* is what we get if we interpret abstract definition of biadjunction in the three-dimensional structure of dc-categories.

We now state the central theorems.

Theorem 1: If $(F, U, \eta, \varepsilon, \mu, \nu)$ is a semi-lax adjunction, then U is a pseudofunctor.

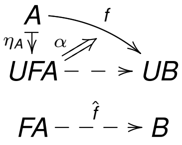
This is remarkable, as it reveals an asymmetry in the concept of semi-lax adjunction.

Semi-lax adjunctions

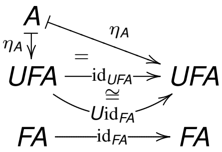
Theorem 2: Let $\mathfrak{A}, \mathfrak{B}$ be dc-categories and let $(U, \phi) : \mathfrak{B} \to \mathfrak{A}$ be a pseudo functor that maps regular arrows to regular arrows. Then U has a left semi-lax adjoint *iff*

1. For each $A \in \mathrm{Obj}(\mathfrak{A})$ there is an $FA \in \mathrm{Obj}(\mathfrak{B})$ and a regular arrow $\eta_A : A \mapsto UFA$ such that for all $B \in \mathrm{Obj}(\mathfrak{B})$ and $f : A \to UB$, the category $(A \nearrow U)(\eta_A, f)$ has a terminal object $(\hat{f}, \alpha_f)$.
2. If $f : A \to UB$ is regular then $\hat{f}$ is also regular and α_f is invertible.
3. $(\mathrm{id}_{FA}, \phi_{FA}^{-1} \circ \eta_A)$ is terminal in $(A \nearrow U)(\eta_A, \eta_A)$.
4. For all $f : A \to UB$ and all *regular* $g : B \mapsto C$, $(g\hat{f}, (\phi_{(\hat{f},g)}^{-1} \circ \eta_A)(Ug \circ \alpha_f))$ is terminal in $(A \nearrow U)(\eta_A, Ugf)$.

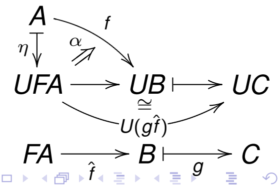
Ad (1):



Ad (3):



Ad (4):



Semi-lax adjunctions

Theorem 3: The forgetful functor $\boldsymbol{S}$ from toposes to triposes has a semi-lax left adjoint.

Conclusion:

What have we achieved?

- We found a universal characterization of the tripos-to-topos construction.
- We found an interesting tricategory(?) with lax features.