A decomposition of the tripos-to-topos construction

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Part 1

A universal characterization of the tripos-to-topos construction

A universal characterization of the tripos-to-topos construction

- What should a universal characterization of the tripos-to-topos construction look like?
- It should be something two-dimensional, since triposes and toposes form 2-categories in a natural way.

Definition of Tripos

Let $\mathbb{C}$ be a category with finite limits. A tripos over $\mathbb{C}$ is a functor

$$\mathcal{P} : \mathbb{C}^{op} \to \textbf{Poset},$$

such that

1. For each $A \in \mathbb{C}$, $\mathcal{P}(A)$ is a Heyting algebra[1].
2. For all $f : A \to B$ in $\mathbb{C}$ the maps $\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ preserve all structure of Heyting algebras.
3. For all $f : A \to B$ in $\mathbb{C}$, the maps $\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ have left and right adjoints
$$\exists_f \dashv \mathcal{P}(f) \dashv \forall_f$$
subject to the Beck-Chevalley condition.
4. For each $A \in \mathbb{C}$ there exists $\pi A \in \mathbb{C}$ and $(\ni_A) \in \mathcal{P}(\pi A \times A)$ such that for all $\psi \in \mathcal{P}(C \times A)$ there exists $\chi_\psi : C \to \pi A$ such that
$$\mathcal{P}(\chi_\psi \times A)(\ni_A) = \psi.$$

[1] A Heyting algebra is a poset which is bicartesian closed as a category.

Tripos morphisms

A tripos morphism between triposes $\mathcal{P} : \mathbb{C}^{\text{op}} \to \textbf{Poset}$ and $\mathcal{Q} : \mathbb{D}^{\text{op}} \to \textbf{Poset}$ is a pair (F, Φ) of a functor

$$F \ : \ \mathbb{C} \to \mathbb{D}$$

and a natural transformation

$$\Phi \ : \ \mathcal{P} \to \mathcal{Q} \circ F$$

such that

1. F preserves finite products
2. For every $C \in \mathbb{C}$, Φ_C preserves finite meets.

If Φ commutes with existential quantification, i.e.

$$\Phi_D(\exists_f \psi) = \exists_{Ff} \Phi_C(\psi)$$

for all $f : C \to D$ in $\mathbb{C}$ and $\psi \in \mathcal{P}(C)$, then we call the tripos morphism regular.

Tripos transformations

A tripos transformation

$$\eta : (F, \Phi) \to (G, \Gamma) : \mathcal{P} \to \mathcal{Q}$$

is a natural transformation

$$\eta : F \to G$$

such that for all $C \in \mathbb{C}$ and all $\psi \in \mathcal{P}(C)$, we have

$$\Phi_C(\psi) \ \leq \ \mathcal{Q}(\eta_C)(\Gamma_C(\psi)).$$

The 2-category **Trip** of triposes

Triposes, tripos morphisms and tripos transformations form a 2-category which we call **Trip**.

The 2-category **Top** of toposes

Toposes, finite limit preserving functors and arbitrary natural transformations form a 2-category which we call **Top**.

The functor S : Top $\to$ Trip

- For a given topos $\mathcal{E}$, the functor $\mathcal{E}(-, \Omega)$ is a tripos if we equip the homsets with the inclusion ordering of the classified subobjects
- This construction is 2-functorial and gives rise to a 2-functor

$$S : \textbf{Top} \to \textbf{Trip}$$

- The tripos-to-topos construction can't be a left biadjoint of $\boldsymbol{S}$, since it is oplax functorial (examples later).
- However, there *is* a characterization as a generalized biadjunction.

Dc-categories

Definition

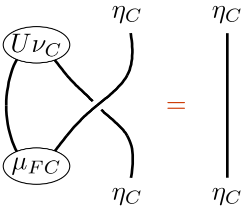
1. A dc-category is given by a 2-category $\mathscr{C}$ together with a designated subclass $\mathscr{C}_r$ of the class of all 1-cells which contains identities and is closed under composition and vertical isomorphisms.
   Elements of $\mathscr{C}_r$ are called regular 1-cells.
   We call a dc-category geometric, if all left adjoints in it are regular.
2. A special functor between dc-categories $\mathscr{C}$ and $\mathscr{D}$ is an oplax functor $F : \mathscr{C} \to \mathscr{D}$ such that Ff is a regular 1-cell whenever f is a regular 1-cell, all identity constraints $FI_A \to I_{FA}$ are invertible, and the composition constraints $F(gf) \to Fg\,Ff$ are invertible whenever g is a regular 1-cell.
3. A special transformation between special functors F, G is an oplax natural transformation $\eta : F \to G$ such that all η_A are regular 1-cells and the naturality constraint $\eta_B\,Ff \to Gf\,\eta_A$ is invertible whenever f is a regular 1-cell.

Special biadjunctions

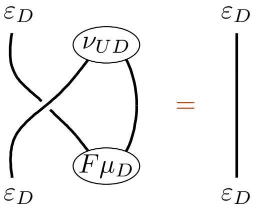
A special biadjunction between dc-categories $\mathscr{C}$ and $\mathscr{D}$ is given by

- special functors $F : \mathscr{C} \to \mathscr{D}$ $\quad U : \mathscr{D} \to \mathscr{C}$,
- special transformations $\eta : \mathrm{id}_{\mathscr{C}} \to UF$ $\quad \varepsilon : FU \to \mathrm{id}_{\mathscr{D}}$
- invertible modifications $\mu : \mathrm{id}_U \to U\varepsilon \circ \eta U$ $\quad \nu : \varepsilon F \circ F\eta \to \mathrm{id}_F$

such that the equalities



and



hold for all $C \in \mathscr{C}$ and $D \in \mathscr{D}$.

Properties of special biadjunctions

- If they exist, special biadjoints are unique up to equivalence.
- For any special biadjunction $F \dashv U$, the right adjoint U is strong.

The dc-categories of triposes and toposes

- To give **Top** and **Trip** the structure of dc-categories, specify classes of *regular* 1-cells.
- A regular 1-cell in **Trip** is a tripos morphism which commutes with $\exists$.
- A regular 1-cell in **Top** is a functor which preserves epimorphisms (besides finite limits).

The characterization

Theorem

The 2-functor $S : \mathbf{Top} \to \mathbf{Trip}$ *is a special functor and has a special left biadjoint*

$$\boldsymbol{T} \dashv \boldsymbol{S} : \mathbf{Top} \to \mathbf{Trip}$$

whose object part is the tripos-to-topos construction.

The topos $T\mathcal{P}$

For a tripos $\mathcal{P}$ on $\mathbb{C}$, $T\mathcal{P}$ is given as follows:

- The objects of $T\mathcal{P}$ are pairs $A = (|A|, \sim_A)$, where $|A| \in \mathrm{obj}(\mathbb{C})$, $(\sim_A) \ \in \mathcal{P}(|A| \times |A|)$, and the judgments

$$x \sim_A y \vdash y \sim_A x$$
$$x \sim_A y, y \sim_A z \vdash x \sim_A z$$

  hold in the logic of $\mathcal{P}$.

  Intuition: "$\sim_A$ is a partial equivalence relation on $|A|$ in the logic of $\mathcal{P}$"

The topos $\boldsymbol{T}\mathcal{P}$

- A morphism from A to B is a predicate $\phi \in \mathcal{P}(|A| \times |B|)$ such that the following judgments hold in $\mathcal{P}$.

(strict)	$\phi(x, y) \vdash x \sim_A x \wedge y \sim_B y$
(cong)	$\phi(x, y), x \sim_A x', y \sim_B y' \vdash \phi(x', y')$
(singval)	$\phi(x, y), \phi(x, y') \vdash y \sim_B y'$
(tot)	$x \sim_A x \vdash \exists y . \phi(x, y)$

The topos $T\mathcal{P}$

- The composition of two morphisms

$$A \xrightarrow{\phi} B \xrightarrow{\gamma} C,$$

  is given by

$$(\gamma \circ \phi)(a, c) \quad \equiv \quad \exists b \,.\, \phi(a, b) \wedge \gamma(b, c).$$

- The identity morphism on A is $\sim_A$.

Mapping tripos morphisms to functors between toposes

Given a regular tripos morphism

$$(F, \Phi) : \mathcal{P} \to \mathcal{Q},$$

we can define a functor

$$\boldsymbol{T}(F, \Phi) : \boldsymbol{T}\mathcal{P} \to \boldsymbol{T}\mathcal{Q}$$

by

$$\begin{array}{ccc} (|A|, \sim_A) & \mapsto & (F(|A|), \Phi(\sim_A)) \\ (\gamma : (|A|, \sim_A) \to (|B|, \sim_B)) & \mapsto & \Phi\gamma \end{array}$$

This works because the definition of partial equivalence relations, functional relations and composition only uses $\wedge$ and $\exists$, which are preserved by regular tripos morphisms.

Mapping tripos morphisms to functors between toposes

- This method only works if (F, Φ) is regular.
- For plain tripos morphisms, we have to use a trick involving *weakly complete objects*.

Weakly complete objects

Definition

(C,τ) in $\boldsymbol{T}\mathcal{P}$ is *weakly complete*, if for every

$$\phi : (A,\rho) \to (C,\tau),$$

there exists a morphism $f : A \to C$ (in the base category) such that

$$\phi(a,c) \dashv\vdash \rho(a,a) \wedge \tau(fa,c)$$

- f is not unique, but ϕ can be reconstructed from f.
- For weakly complete (C,τ), $\boldsymbol{T}\mathcal{P}((A,\rho),(C,\sigma))$ is a quotient of $\mathbb{C}(A,C)$ by the partial equivalence relation

$$f \sim g \quad \Leftrightarrow \quad \rho(x,y) \vdash \sigma(fx,gy).$$

Weakly complete objects (continued)

- For each object (A, ρ) in $\boldsymbol{T}\mathcal{P}$, there is an isomorphic weakly complete object $(\tilde{A}, \tilde{\rho})$ with underlying object πA and partial equivalence relation

$$
\begin{aligned}
m, n{:}\pi(A) \mid (\exists x{:}A\,.\,\rho(x,x) \wedge \forall y{:}A\,.\,y \in m \Leftrightarrow \rho(x,y)) \\
\wedge(\forall x\,.\,x \in m \Leftrightarrow x \in n)
\end{aligned}
$$

- This means that $\boldsymbol{T}\mathcal{P}$ is equivalent to its full subcategory $\widetilde{\boldsymbol{T}\mathcal{P}}$ on the weakly complete objects.
- For an *arbitrary* tripos morphism $(F, \Phi) : \mathcal{P} \to \mathcal{R}$, we can define a functor

$$\tilde{\boldsymbol{T}}(F, \Phi) : \widetilde{\boldsymbol{T}\mathcal{P}} \to \boldsymbol{T}\mathcal{R}$$

by

$$
\begin{array}{ccc}
(A, \rho) & \mapsto & (FA, \Phi\rho) \\
\downarrow [f] & \mapsto & \downarrow (a, b \mid \rho(a,a) \wedge \sigma(Ffa, b)) \\
(B, \sigma) & \mapsto & (FB, \Phi\rho)
\end{array}
$$

- Problem: In general we have to pre- or postcompose by the equivalence $\boldsymbol{T}\mathcal{P} \simeq \widetilde{\boldsymbol{T}\mathcal{P}}$, which renders computations complicated.
- Role of weakly complete objects conceptually not clear.
- Proposed solution: decompose the tripos-to-topos construction in two steps, in the intermediate step, the weakly complete objects have a categorical characterization.

Part 2

A decomposition of the tripos-to-topos construction

The category $\boldsymbol{F}\mathcal{P}$

Definition

For a tripos $\mathcal{P}$ we define a category $\boldsymbol{F}\mathcal{P}$ such that

- $\boldsymbol{F}\mathcal{P}$ has the same objects as $\boldsymbol{T}\mathcal{P}$
- $\boldsymbol{F}\mathcal{P}((A,\rho),(B,\sigma))$ is the subquotient of $\mathbb{C}(A,B)$ by

$$f \sim g \quad \Leftrightarrow \quad \rho(x,y) \vdash \sigma(fx,gy).$$

- $\boldsymbol{F}\mathcal{P}$ can be identified with a *luff* subcategory of $\boldsymbol{T}\mathcal{P}$.

Coarse objects

- **Central observation:** Weakly complete objects in $\boldsymbol{T}\mathcal{P}$ can be characterized as *coarse objects* in $\boldsymbol{F}\mathcal{P}$, where *coarse* is defined as follows.

Definition

An object C of a category is called coarse, if for every morphism $f : A \rightarrowtail\!\!\!\!\rightarrow B$ which is monic and epic at the same time, and every $g : A \to C$ there exists a mediating arrow in

$$\begin{array}{ccc} A & \overset{f}{\rightarrowtail\!\!\!\!\twoheadrightarrow} & B \\ & {\scriptstyle g}\searrow & \vdots \\ & & C \end{array}.$$

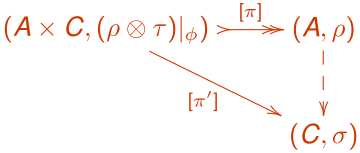
Coarse objects

Lemma

Weakly complete objects in $\boldsymbol{T}\mathcal{P}$ coincide with coarse objects in $\boldsymbol{F}\mathcal{P}$.

Proof:

- Weakly complete objects are coarse, because mono-epis in $\boldsymbol{F}\mathcal{P}$ are isos in $\boldsymbol{T}\mathcal{P}$.
- To see that coarse objects are weakly complete, let $\phi : (A, \rho) \to (C, \tau)$ in $\boldsymbol{T}\mathcal{P}$, and consider the following diagram in $\boldsymbol{F}\mathcal{P}$:



The mediator gives the desired morphism in the base.

2nd observation: The coarse objects of $\boldsymbol{F}\mathcal{P}$ form a reflective subcategory (which we will call $\boldsymbol{T}\mathcal{P}$ from now on).

$$J \dashv I : \boldsymbol{T}\mathcal{P} \to \boldsymbol{F}\mathcal{P}$$

Given an arbitrary tripos morphism $(F, \Phi) : \mathcal{P} \to \mathcal{R}$, we can now define

$$\begin{array}{rcrcl} \boldsymbol{F}(F,\Phi) & : & \boldsymbol{F}\mathcal{P} & \to & \boldsymbol{F}\mathcal{R} \\ & & (A,\rho) & \mapsto & (FA, \Phi\rho) \\ & & [f] & \mapsto & [Ff] \end{array}$$

and we obtain a a functor between $\boldsymbol{T}\mathcal{P}$ and $\boldsymbol{T}\mathcal{Q}$ by pre- and postcomposing by the right and left adjoints of the reflections.

An abstract look at the decomposition

Abstractly, the decomposition arises when we factor the forgetful functor $S : \mathbf{Top} \to \mathbf{Trip}$ through an intermediate dc-category

$$\begin{array}{ccc} \mathbf{Top} & \xrightarrow{\quad S \quad} & \mathbf{Trip} \\ & {\scriptstyle U}\searrow \quad \nearrow{\scriptstyle S} & \\ & \mathbf{QTop} & \end{array} ,$$

the dc-category of q-toposes.

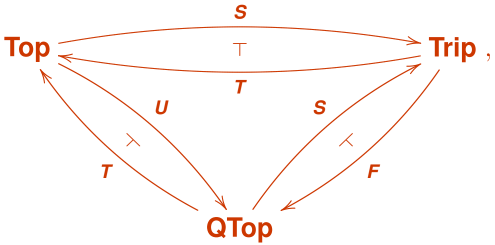
Q-Toposes

Definition

- A monomorphism $m : U \to B$ in a category $\mathcal{C}$ is called strong, if for every commutative square

$$\begin{array}{ccc} A & \longrightarrow & U \\ {\scriptstyle e}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle m} \\ Q & \longrightarrow & B \end{array}$$

  where e is an epimorphism, there exists a (unique) h.
- A q-topos is a category $\mathcal{C}$ with finite limits, an exponentiable classifier of strong monomorphisms, and pullback stable quotients of strong equivalence relations.
- The dc-category of q-toposes has finite limit preserving functors as 1-cells. Regular 1-cells additionally preserve epimorphisms *and* strong epimorphisms.



We have to prove that

- The presheaf $\boldsymbol{S}\mathcal{C}$ of strong subobjects of a q-topos $\mathcal{C}$ is a tripos.
- For any tripos $\mathcal{P}$, the category $\boldsymbol{F}\mathcal{P}$ is a q-topos.
- The coarse objects of any q-topos form a reflective subcategory which is a topos.

Q-toposes to triposes

To show that the presheaf of strong monomorphisms on a q-topos is a tripos, we define an internal language which is very similar to the type theory based on equality in the book *Higher order categorical logic* of Lambek and Scott.

Types:

$$A ::= X \mid 1 \mid \Omega \mid PA \mid A \times A \qquad X \in \mathrm{obj}(\mathcal{C})$$

Terms:

We use Δ to denote a context $x_1{:}A_1, \ldots, x_n{:}A_n$ of typed variables.

$$\frac{}{\Delta \mid x_i : A_i}\ (i=1,\ldots,n) \qquad \frac{}{\Delta \mid * : 1}$$

$$\frac{\Delta, x{:}A \mid \varphi[x] : \Omega}{\Delta \mid \{x|\varphi[x]\} : PA} \qquad \frac{\Delta \mid a : A \qquad \Delta \mid b : B}{\Delta \mid (a, b) : A \times B}$$

$$\frac{\Delta \vdash a : A \qquad \Delta \vdash M : PA}{\Delta \vdash a \in M : \Omega} \qquad \frac{\Delta \vdash a : A \qquad \Delta \vdash a' : A}{\Delta \vdash a = a' : \Omega}$$

$$\frac{\Delta \mid a : X}{\Delta \mid f(a) : Y}\ f \in \mathcal{C}(X, Y)$$

Deduction rules:

$$\frac{}{\Delta \mid p_1, \ldots, p_n \vdash p_i}\ \text{Ax}\ (i=1,\ldots,n) \qquad \frac{\Delta \mid \Gamma \vdash p \qquad \Delta \mid \Gamma, p \vdash q}{\Delta \mid \Gamma \vdash q}\ \text{Cut}$$

$$\frac{}{\Delta \mid \Gamma \vdash t = t}\ {=}\text{R} \qquad \frac{\Delta, x{:}A \mid \Gamma \vdash \varphi[x, x]}{\Delta \mid \Gamma, s = t \vdash \varphi[s, t]}\ {=}\text{L}$$

$$\frac{\Delta, x{:}A \mid \Gamma \vdash p[x] = (x \in M)}{\Delta \mid \Gamma \vdash \{x|p[x]\} = M}\ \text{P-}\eta \qquad \frac{}{\Delta \mid \Gamma \vdash (a \in \{x|p[x]\}) = p[a]}\ \text{P-}\beta$$

$$\frac{}{\Delta \mid \Gamma \vdash t = *}\ 1\text{-}\eta \qquad \frac{\Delta \mid \Gamma, p \vdash q \qquad \Delta \mid \Gamma, q \vdash p}{\Delta \mid \Gamma \vdash p = q}\ \text{Ext}$$

Q-toposes to toposes

To obtain the coarse reflection $\overline{C}$ of an object C of a q-topos $\mathcal{C}$, we take the epi / strong mono factorization of the canonical mono $C \rightarrowtail PC$.

$$C \twoheadrightarrow \overline{C} \rightarrowtriangle PC$$

Since coarse objects are closed under finite limits, and the power objects are already coarse, it follows that the subcategory is a topos.

Triposes to q-toposes

left out

Part 3

Examples

Triposes from complete Heyting algebras

- For a complete Heyting algebra A, the functor

$$\mathcal{P}_A \;=\; \mathbf{Set}(-, A)$$

  is a tripos if we equip the sets $\mathbf{Set}(I, A)$ with the pointwise ordering.
- For a meet preserving map $f : A \to A'$ between complete Heyting algebras, the induced natural transformation

$$\mathcal{P}_f = \mathbf{Set}(-, f) \;:\; \mathbf{Set}(-, A) \to \mathbf{Set}(-, A')$$

  is a tripos morphism
- $\boldsymbol{F}\mathcal{P}_A \simeq \mathbf{Sep}(A)$ (separated presheaves on A)
- $\boldsymbol{T}\mathcal{P}_A \simeq \mathbf{Sh}(A)$ (sheaves on A)

Example

- $\mathbb{B}$ is the 2-element Heyting algebra $\mathbb{B} = \{\text{true}, \text{false}\}$ with false $\leq$ true.
- $$\mathbb{B} \xrightarrow{\quad\delta\quad} \mathbb{B} \times \mathbb{B} \xrightarrow{\quad\wedge\quad} \mathbb{B}$$

Example

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- $$\mathbb{B} \xrightarrow{\ \delta\ } \mathbb{B} \times \mathbb{B} \xrightarrow{\ \wedge\ } \mathbb{B}$$

  $$\mathcal{P}_{\mathbb{B}} \xrightarrow{\ \mathcal{P}_\delta\ } \mathcal{P}_{\mathbb{B}\times\mathbb{B}} \xrightarrow{\ \mathcal{P}_\wedge\ } \mathcal{P}_{\mathbb{B}}$$

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$$\mathcal{P}_{\mathbb{B}} \xrightarrow{\mathcal{P}_{\delta}} \mathcal{P}_{\mathbb{B}\times\mathbb{B}} \xrightarrow{\mathcal{P}_{\wedge}} \mathcal{P}_{\mathbb{B}}$$

$$\mathbf{Sep}(\mathbb{B}) \longrightarrow \mathbf{Sep}(\mathbb{B} \times \mathbb{B}) \longrightarrow \mathbf{Sep}(\mathbb{B})$$

Example

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-

$$\mathbb{B} \xrightarrow{\quad\delta\quad} \mathbb{B} \times \mathbb{B} \xrightarrow{\quad\wedge\quad} \mathbb{B}$$

$$\mathcal{P}_{\mathbb{B}} \xrightarrow{\quad\mathcal{P}_{\delta}\quad} \mathcal{P}_{\mathbb{B}\times\mathbb{B}} \xrightarrow{\quad\mathcal{P}_{\wedge}\quad} \mathcal{P}_{\mathbb{B}}$$

$$\begin{array}{ccccc}
\mathbf{Sep}(\mathbb{B}) & \longrightarrow & \mathbf{Sep}(\mathbb{B}\times\mathbb{B}) & \longrightarrow & \mathbf{Sep}(\mathbb{B}) \\
\downarrow\uparrow \dashv & & \downarrow\uparrow \dashv & & \downarrow\uparrow \dashv \\
\mathbf{Sh}(\mathbb{B}) \simeq \mathbf{Set} & & \mathbf{Sh}(\mathbb{B}\times\mathbb{B}) \simeq \mathbf{Set}\times\mathbf{Set} & & \mathbf{Sh}(\mathbb{B}) \simeq \mathbf{Set}
\end{array}$$

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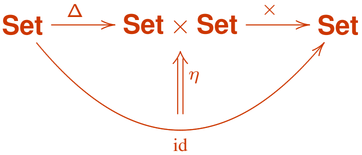
  $$\mathcal{P}_{\mathbb{B}} \xrightarrow{\ \mathcal{P}_{\delta}\ } \mathcal{P}_{\mathbb{B}\times\mathbb{B}} \xrightarrow{\ \mathcal{P}_{\wedge}\ } \mathcal{P}_{\mathbb{B}}$$

  $$\begin{array}{ccccc}
  \mathbf{Sep}(\mathbb{B}) & \longrightarrow & \mathbf{Sep}(\mathbb{B} \times \mathbb{B}) & \longrightarrow & \mathbf{Sep}(\mathbb{B}) \\
  \downarrow \dashv \uparrow & & \downarrow \dashv \uparrow & & \downarrow \dashv \uparrow \\
  \mathbf{Sh}(\mathbb{B}) \simeq \mathbf{Set} & \xrightarrow[\Delta]{} & \mathbf{Sh}(\mathbb{B} \times \mathbb{B}) \simeq \mathbf{Set} \times \mathbf{Set} & \xrightarrow[\times]{} & \mathbf{Sh}(\mathbb{B}) \simeq \mathbf{Set}
  \end{array}$$

Example

- Comparing the composition of the images of the tripos transformations with the image of the composition we get



- This shows that the tripos-to-topos construction is only oplax functorial, as claimed earlier.

Analyzing the unit of $T \dashv S$

The unit of $T \dashv S$: **Top** $\to$ **Trip** gives rise to 1-cells $(D, \Delta) : \mathcal{P} \to ST\mathcal{P}$

and to 2-cells
$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{(F,\Phi)} & \mathcal{R} \\ \downarrow & \Downarrow & \downarrow \\ ST\mathcal{P} & \longrightarrow & ST\mathcal{R} \end{array}$$
which decompose into
$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{(F,\Phi)} & \mathcal{R} \\ \downarrow & \Downarrow\alpha & \downarrow \\ SF\mathcal{P} & \longrightarrow & SF\mathcal{R} \\ \downarrow & \Downarrow\beta & \downarrow \\ ST\mathcal{P} & \longrightarrow & ST\mathcal{R} \end{array}.$$

Lemma

α is an isomorphism whenever Φ commutes with $\exists$ along diagonal mappings $\delta : A \to A \times A$, and β is an isomorphism whenever Φ commutes with $\exists$ along projections. Furthermore, α is always an epimorphism and β is always a monomorphism.

Example

The tripos transformation $\mathcal{P}_\wedge : \mathcal{P}_{\mathbb{B}\times\mathbb{B}} \to \mathcal{P}_{\mathbb{B}}$ commutes with $\exists$ along δ. Therefore we have

$$
\begin{array}{ccc}
\textbf{Set} & \xrightarrow{\mathrm{id}} & \textbf{Set} \\
\downarrow & \cong & \downarrow \\
\textbf{Sep}(\mathbb{B}\times\mathbb{B}) & \longrightarrow & \textbf{Sep}(\mathbb{B}) \\
\downarrow & \Downarrow\beta & \downarrow \\
\textbf{Set}\times\textbf{Set} & \longrightarrow & \textbf{Set}
\end{array}
\;.
$$

Example: Modified realizability

The embedding

$$\nabla = (\neg\neg \circ \Delta) \quad : \quad \mathcal{P}_{\mathbb{B}} \to \mathbf{mr}$$

of the classical predicates into the modified realizability tripos **mr** commutes with $\exists$ along projections. This gives

$$\begin{array}{ccc}
\mathbf{Set} & \xrightarrow{\mathrm{id}} & \mathbf{Set} \\
\downarrow & \Downarrow\alpha & \downarrow \\
\boldsymbol{F}(\mathcal{P}_{\mathbb{B}}) & \longrightarrow & \boldsymbol{F}(\mathbf{mr}) \\
\downarrow & \cong & \downarrow \\
\mathbf{Set} & \longrightarrow & \boldsymbol{T}(\mathbf{mr})
\end{array} .$$