Krivine's Classical Realizability from a Categorical Prespective

Thomas Streicher (TU Darmstadt)

July 2010

The Scenario

In Krivine's work on **Classical Realizability** he emphasizes that his notion of realizability is a **generalization of forcing** as known from set theory.

Thus Krivine's classical realizability is not captured by partial combinatory algebras (pca's) as known from realizability (toposes) since

$$\mathrm{RT}(\mathbb{A}) \text{ Groth. topos} \quad \Rightarrow \quad \mathbb{A} \text{ trivial pca}$$

But the **order pca**'s of J. van Oosten and P. Hofstra provide a common generalization of realizability and Heyting valued models.

Classical Realizability (1)

The collection of (possibly open) terms is given by the grammar

$$t ::= x \mid \lambda x.t \mid ts \mid \mathsf{cc}\, t \mid \mathsf{k}_\pi$$

where π ranges over stacks (i.e. lists) of closed terms. We write Λ for the set of closed terms and Π for the set of stacks of closed terms.

A *process* is a pair $t * \pi$ with $t \in \Lambda$ and $\pi \in \Pi$.

The operational semantics of Λ is given by the relation $\succeq$ (*head reduction*) on processes defined inductively by the clauses

(pop)	$\lambda x.t * s.\pi$	$\succeq$	$t[s/x] * \pi$
(push)	$ts * \pi$	$\succeq$	$t * s.\pi$
(store)	$\mathsf{cc}\, t * \pi$	$\succeq$	$t * \mathsf{k}_\pi.\pi$
(restore)	$\mathsf{k}_\pi * t.\pi'$	$\succeq$	$t * \pi$

Classical Realizability (2)

This language has a natural interpretation within the bifree solution of

$$D \cong \Sigma^{\mathsf{List}(D)} \cong \prod_{n \in \omega} \Sigma^{D^n}$$

NB We have $D \cong \Sigma \times D^D$. Thus D^D is a retract of D and, accordingly, D is a model for λ_β-calculus.

The interpretation of Λ is given by

$$\llbracket \lambda x.t \rrbracket\, \varrho \langle\rangle = \top$$
$$\llbracket \lambda x.t \rrbracket\, \varrho \langle d, k\rangle = \llbracket t \rrbracket\, \varrho[d/x]k$$
$$\llbracket ts \rrbracket\, \varrho k = \llbracket t \rrbracket\, \varrho \langle \llbracket s \rrbracket\, \varrho, k\rangle$$
$$\llbracket \mathsf{cc}\, t \rrbracket\, \varrho k = \llbracket t \rrbracket\, \varrho \langle \mathsf{ret}(k), k\rangle$$
$$\llbracket \mathsf{k}_\pi \rrbracket\, \varrho = \mathsf{ret}(\llbracket \pi \rrbracket\, \varrho)$$

where

$$\mathsf{ret}(k)\langle\rangle = \top$$
$$\mathsf{ret}(k)\langle d, k'\rangle = d(k)$$

and

$$\llbracket \langle\rangle \rrbracket\, \varrho = \langle\rangle$$
$$\llbracket t.\pi \rrbracket\, \varrho = \langle \llbracket t \rrbracket\, \varrho, \llbracket \pi \rrbracket\, \varrho\rangle$$

Classical Realizability (3)

A set $\perp\!\!\!\perp$ of processes is called *saturated* iff $q \in \perp\!\!\!\perp$ whenever $q \succeq p \in \perp\!\!\!\perp$. We write $t \perp \pi$ for $t{*}\pi \in \perp\!\!\!\perp$. (In the model D one may choose $\perp\!\!\!\perp$ as an arbitrary subset of $D \times \mathsf{List}(D)$, *e.g.* $\perp\!\!\!\perp = \{t * \pi \mid t(\pi) = \top\}$.)
For $X \subseteq \Pi$ and $Y \subseteq \Lambda$ we put

$$X^\perp = \{t \in \Lambda \mid \forall \pi \in X.\, t \perp \pi\}$$
$$Y^\perp = \{\pi \in \Pi \mid \forall t \in Y.\, t \perp \pi\}$$

Obviously $(-)^\perp$ is antitonic and $Z \subseteq Z^{\perp\perp}$ and thus $Z^\perp = Z^{\perp\perp\perp}$.
For a saturated set $\perp\!\!\!\perp$ of processes second order logic over a set M of individuals is interpreted as follows: n-ary predicate variables range over functions $M^n \to \mathcal{P}(\Pi)$ and formulas A are interpreted as $||A|| \subseteq \Pi$

$$||X(t_1, \ldots, t_n)||_\varrho = \varrho(X)([\![t_1]\!]_\varrho, \ldots, [\![t_1]\!]_\varrho)$$
$$||A{\to}B||_\varrho = |A|_\varrho . ||B||_\varrho$$
$$||\forall x A(x)|| = \bigcup_{a\in M} ||A(a)||$$
$$||\forall X A[X]||_\varrho = \bigcup_{R\in\mathcal{P}(\Pi)^{M^n}} ||A||_{\varrho[R/X]}$$

where $|A|_\varrho = ||A||^\perp_\varrho$.

Classical Realizability (4)

We have $|\forall X A| = \bigcap_{R \in \mathcal{P}(\Pi)^{M^n}} |A[R/X]|$.

In general $|A{\to}B|$ is a **proper** subset of

$$|A|{\to}|B| = \{t{\in}\Lambda \mid \forall s{\in}|A|\ ts \in |B|\}$$

since in general

$$ts * \pi \in \perp\!\!\!\perp \not\Rightarrow t * s.\pi \in \perp\!\!\!\perp$$

But for every $t \in |A|{\to}|B|$ its η-expansion $\lambda x.tx \in |A{\to}B|$.

But, of course, we have $|A{\to}B| = |A|{\to}|B|$ whenever $\perp\!\!\!\perp$ is also *closed under head reduction*, i.e. $\perp\!\!\!\perp \ni p \succeq q$ implies $q \in \perp\!\!\!\perp$.

One may even assume that $\perp\!\!\!\perp$ is stable w.r.t. the semantic equality $=_D$ induced by the model D. In particular $\Lambda_{/=_D}$ is a pca.

Classical Realizability (5)

However, there are interesting situations where one has to go beyond such a framework. For realizing the countable choice axiom CAC Krivine introduced a new language construct χ^* with the reduction rule

$$\chi^* * t.\pi \succeq t * n_t.\pi$$

where n_t is the Church numeral representation of a Gödel number for t, *c.f.* $\mathsf{quote}(t)$ of LISP.

NB quote is in conflict with β-reduction!

NB The term χ^* realizes *Krivine's Axiom*

$$\exists S \forall x \Big(\forall n^{\mathsf{Int}} Z(x, S_{x,n}) \to \forall X Z(x, X) \Big)$$

which entails CAC.

Axiomatic Class. Realiz. (1)

Instead of the usual pca's one may consider the following axiomatic framework which we call **Abstract Krivine Structure** (AKS) :

- a set Λ of "terms" together with a binary application operation (written as juxtaposition) and distinguished elements K, $\mathsf{S}, \mathsf{cc} \in \Lambda$

- a set Π of "stacks" together with a push operation (push) from $\Lambda \times \Pi$ to Π (written $t.\pi$) and a unary operation $\mathsf{k} : \Pi \to \Lambda$

- a saturated subset $\perp\!\!\!\perp$ of $\Lambda \times \Pi$

where *saturated* means that $\perp\!\!\!\perp^c = \Lambda \times \Pi \setminus \perp\!\!\!\perp$ satisfies the closure conditions

(S1) $ts \star \pi$ in $\perp\!\!\!\perp^c$ implies $t \star s.\pi$ in $\perp\!\!\!\perp^c$

(S2) $\mathsf{K} \star t.s.\pi$ in $\perp\!\!\!\perp^c$ implies $t \star \pi$ in $\perp\!\!\!\perp^c$

(S3) $\mathsf{S} \star t.s.u.\pi$ in $\perp\!\!\!\perp^c$ implies $tu(su) \star \pi$ in $\perp\!\!\!\perp^c$

(S4) $\mathsf{cc} \star t.\pi$ in $\perp\!\!\!\perp^c$ implies $t \star \mathsf{k}_\pi.\pi$ in $\perp\!\!\!\perp^c$

(S5) $\mathsf{k}_\pi \star t.\pi'$ in $\perp\!\!\!\perp^c$ implies $t \star \pi$ in $\perp\!\!\!\perp^c$.

Axiomatic Class. Realiz. (2)

A *proposition* A is given by a subset $||A|| \subseteq \Pi$. The set of *realizers* for A is given by

$$|A| = ||A||^{\perp} = \{t \in \Lambda \mid \forall \pi \in ||A|| \;\; t \star \pi \in \perp\!\!\!\perp\}$$

Logic is interpreted as follows

$$||R(\vec{t})|| = R\Big(\llbracket \vec{t} \rrbracket\Big)$$

$$||A \to B|| = |A|.||B|| = \{t.\pi \mid t \in |A|, \pi \in ||B||\}$$

$$||\forall x A(x)|| = \bigcup_{a \in M} ||A(a)||$$

$$||\forall X A(X)|| = \bigcup_{R \in \mathcal{P}(\Pi)^{M^n}} ||A(R)||$$

where M is the underlying set of the model.

NB On could define propositions more restrictively as

$$\mathcal{P}_{\perp\!\!\!\perp}(\Pi) = \{X \in \mathcal{P}(\Pi) \mid X = X^{\perp\perp}\}$$

and this would not change the meaning of $|A|$ for closed formulas (though it would change the meaning of $||A||$).

Axiomatic Class Realiz. (3)

Notice that $\mathcal{P}_{\perp\!\!\perp}(\Pi)$ is in 1-1-correspond. with

$$\mathcal{P}_{\perp\!\!\perp}(\Lambda) = \{X \in \mathcal{P}(\Lambda) \mid X = X^{\perp\perp}\}$$

via $(-)^{\perp}$. Then in case (S1) holds as an equivalence, i.e. we have

(SS1) $\quad ts \star \pi$ in $\perp\!\!\perp^c \quad$ iff $\quad t \star s.\pi$ in $\perp\!\!\perp^c$

then one may define $|\cdot|$ directly as

$$|R(\vec{t})| = R\Big(\llbracket \vec{t} \rrbracket\Big)$$

$$|A{\to}B| = |A|{\to}|B| = \{t \in L \mid \forall s \in |A|\ ts \in |B|\}$$

$$|\forall x A(x)| = \bigcap_{a \in M} |A(a)|$$

$$|\forall X A(X)| = \bigcap_{R \in \mathcal{P}_{\perp\!\!\perp}(\Lambda)^{M^n}} |A(R)|$$

and it coincides with the previous definition for closed formulas.

Abstract Krivine structures validating the reasonable assumption (SS1) are called **strong abstract Krivine structures** (SAKSs).

Axiomatic Class Realiz. (4)

Obviously, for $A, B \in \mathcal{P}_{\perp\!\!\perp}(\Lambda)$ we have

$$|A{\to}B| \subseteq |A|{\to}|B| = \{t \in \Lambda \forall s \in |A|\ ts \in |B|\}$$

But for any $t \in |A| \to |B|$ we have

$$\mathsf{E}t \in |A{\to}B|$$

where $\mathsf{E} = \mathsf{S}(\mathsf{KI})$ with $\mathsf{I} = \mathsf{SKK}$.

One easily checks that

$$\mathsf{I} * t.\pi \in \perp\!\!\perp^c \ \Rightarrow \ \ t * \pi \in \perp\!\!\perp^c$$

and thus we have

$$\mathsf{E}t * s.\pi \in \perp\!\!\perp^c \ \Rightarrow \ \ ts * \pi \in \perp\!\!\perp^c$$

because

$$\mathsf{E}t * s.\pi \in \perp\!\!\perp^c \ \Rightarrow \mathsf{KI}s(ts) \in \perp\!\!\perp^c \ \Rightarrow$$
$$\mathsf{I} * ts.\pi \in \perp\!\!\perp^c \ \Rightarrow \ \ ts * \pi \in \perp\!\!\perp^c$$

Then for $s \in |A|$, $\pi \in ||B||$ we have $\mathsf{E}t{*}s.\pi \in \perp\!\!\perp$ because $ts * \pi \in \perp\!\!\perp$ since $t \in |A| \to |B|$.
Thus $\mathsf{E}t \in |A{\to}B|$ as desired.

Forcing as an Instance (1)

Let $\mathbb{P}$ a $\wedge$-semilattice (with top element 1) and $\mathcal{D}$ a *downward closed* subset of $\mathbb{P}$.

Such a situation gives rise to a SAKS where

- $\Lambda = \Pi = \mathbb{P}$
- application and the push operation are interpreted as $\wedge$ in $\mathbb{P}$
- k is the identity on $\mathbb{P}$
- the constants K, S and cc are interpreted as 1
- $\perp\!\!\!\perp = \{(p, q) \in \mathbb{P}^2 \mid p \wedge q \in \mathcal{D}\}$.

We write $p \perp q$ for $p * q \in \perp\!\!\!\perp$, i.e. $p \wedge q \in \mathcal{D}$.

NB This is **not** a pca since application $\wedge$ is commutative and associative and thus $a = kab = kba = b$.

Forcing as an Instance (2)

For $X \subseteq \mathbb{P}$ we put

$$X^\perp = \{p \in \mathbb{P} \mid \forall q \in X\; p \wedge q \in \mathcal{D}\}$$

which is downward closed and contains $\mathcal{D}$ as a subset. For downward closed $X \subseteq \mathbb{P}$ with $\mathcal{D} \subseteq X$ we have

$$X^\perp = \{p \in \mathbb{P} \mid \forall q \leq p\, (q \in X \Rightarrow q \in \mathcal{D})\}$$

Thus, for arbitrary $X \subseteq \mathbb{P}$ we have

$$\begin{aligned} X^{\perp\perp} &= \{p \in \mathbb{P} \mid \forall q \leq p\, (q \in X^\perp \Rightarrow q \in \mathcal{D})\} \\ &= \{p \in \mathbb{P} \mid \forall q \leq p\, (q \notin \mathcal{D} \Rightarrow q \notin X^\perp)\} \\ &= \{p \in \mathbb{P} \mid \forall q \leq p\, (q \notin \mathcal{D} \Rightarrow \\ &\qquad\qquad \exists r \leq q\, (q \notin \mathcal{D} \wedge q \in X))\} \end{aligned}$$

as familiar from Cohen forcing.

Further for downward closed $X, Y \subseteq \mathbb{P}$ with $\mathcal{D} \subseteq X, Y$ one can show that

$$\begin{aligned} X \to Y :&= \{p \in \mathbb{P} \mid \forall q \in X\; p \wedge q \in Y\} \\ &= \{p \in \mathbb{P} \mid \forall q \leq p\, (q \in X \Rightarrow q \in Y)\} \end{aligned}$$

and thus

$$Z \subseteq X \to Y \quad \text{iff} \quad Z \cap X \subseteq Y$$

Forcing as an Instance (3)

Propositions are $A \subseteq \mathbb{P}$ with $A = A^{\perp\perp}$ (as in Girard's *phase semantics*). Thus, propositions are in particular downward closed and contain $\mathcal{D}$ as a subset.

We have $X = X^{\perp\perp}$ iff $\mathcal{D} \subseteq X$ and $p \in X \setminus \mathcal{D}$ whenever for all $q \leq p$ with $q \notin \mathcal{D}$ there exists $r \leq q$ with $r \in X \setminus \mathcal{D}$.

In case $\mathcal{D} = \{0\}$ then $\mathbb{P}^{\uparrow} = \mathbb{P} \setminus \{0\}$ is a conditional $\wedge$-semilattice and propositions are in 1-1-correspondence with *regular* subsets A of $\mathbb{P}^{\uparrow}$, i.e. $p \in A$ whenever $\forall q{\leq}p\, \exists r{\leq}q\; r \in A$, the propositions as considered in **Cohen forcing** over $\mathbb{P}^{\uparrow}$.

For propositions A, B we have

$$\begin{array}{lll} p \in A{\to}B & \text{iff} & \forall q \in A\; p \wedge q \in B \\ & \text{iff} & \forall q \leq p\, (q \in A \Rightarrow q \in B) \\ & \text{iff} & p \in (A.B^{\perp})^{\perp} \end{array}$$

and for $\neg A \equiv A{\to}\bot$ (where $\bot$ is $\mathcal{D}$, the least proposition representing *falsity*) we have

$$p \in \neg A \quad \text{iff} \quad \forall q \in A\; p \wedge q \in \mathcal{D} \quad \text{iff} \quad p \in A^{\perp}$$

as in Cohen forcing.

Characterization of Forcing

One can show that a SAKS arises (up to iso) from a downward closed subset of a $\wedge$-semilattice iff

(1) $\mathsf{k} : P \to L$ is a bijection

(2) application is associative, commutative and idempotent and has a neutral element 1

(3) application coincides with the push operation (when identifying L and P via k).

Remark
The downset $\mathcal{D} = \{t \in L \mid (t, 1) \in \perp\!\!\!\perp\}$ (where 1 is considered as element of P via k).

It is in this sense that

forcing = commutative realizability

as Krivine would put it.

AKS's as total OPCAs (1)

Hofstra and van Oosten's notion of **order partial combinatory algebra** (OPCA) generalizes both PCAs and complete Heyting algebras (cHa's). We will show how every AKS can be organised into a total OPCA.

A **total OPCA** is a triple $(\mathbb{A}, \leq, \bullet)$ where $\leq$ is a partial order on $\mathbb{A}$ and $\bullet$ is a binary monotone operation on $\mathbb{A}$ such that there exist $k, s \in \mathbb{A}$ with

$$k \bullet a \bullet b \leq a \qquad s \bullet a \bullet b \bullet c \leq a \bullet c \bullet (b \bullet c)$$

for all $a, b, c \in \mathbb{A}$.

With every AKS we may associate the total OPCA whose underlying set is $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$, where $a \leq b$ iff $a \supseteq b$ and application is defined as

$$a \bullet b = \{\pi \in P \mid \forall t \in |a|, s \in |b| \;\; t * s.\pi \in \perp\!\!\!\perp\}^{\perp\perp}$$

where $|a| = a^{\perp}$.

NB In case of a SAKS we have

$$|a \bullet b| = \{ts \mid t \in |a|, s \in |b|\}^{\perp\perp}$$

AKS's as total OPCAs (2)

For proving our claim we need

Lemma 1
From $a \leq b \to c$ it follows that $a \bullet b \leq c$.

Lemma 2
If $t \in |a|$ and $s \in |b|$ then $ts \in |a \bullet b|$.

One easily shows that $\{\mathsf{K}\}^{\perp\perp}ab \leq a$.
For showing that $\{\mathsf{S}\}^{\perp\perp} \bullet a \bullet b \bullet z \leq a \bullet c \bullet (b \bullet c)$ it suffices by (multiple applications of) Lemma 1 to show that $s \leq a \to b \to c \to (a \bullet c \bullet (b \bullet c))$. It suffices to show that

$$\mathsf{S} \in |a \to b \to c \to (a \bullet c \bullet (b \bullet c))|$$

For this purpose suppose $t \in |a|$, $s \in |b|$, $u \in |c|$ and $\pi \in a \bullet c \bullet (b \bullet c)$. Applying Lemma 2 iteratively we have $tu(su) \in |a \bullet c \bullet (b \bullet c)|$ and thus $tu(su) * \pi \in \perp\!\!\!\perp$. Since $\perp\!\!\!\perp$ is closed under expansion it follows that $\mathsf{S} * t.s.u.\pi \in \perp\!\!\!\perp$ as desired.

AKS's as total OPCAs (3)

A **filter** in a total OPCA $(\mathbb{A}, \leq, \bullet)$ is a subset Φ of $\mathbb{A}$ closed under $\bullet$ and containing (some choice of) k and s (for $\mathbb{A}$).

Examples

(1) If case of a SAKS induced by a down-closed set $\mathcal{D}$ in a $\wedge$-semilattice $\mathbb{P}$ a natural choice of a filter is $\{\mathbb{P}\}$.
(2) $\Phi = \{a \in \mathcal{P}_{\perp\!\!\perp}(\Lambda) \mid |a| \neq \emptyset\}$ is a filter on the total opca $\mathcal{P}_{\perp\!\!\perp}(\Pi)$ by Lemma 2.

Given a total OPCA $\mathbb{A} = (\mathbb{A}, \leq, \bullet)$ and a filter Φ in $\mathbb{A}$ one may asscoiate with it a $\mathbf{Set}$-indexed preorder $[-, \mathbb{A}]_\Phi$ as follows

- $[I, \mathbb{A}]_\Phi = \mathbb{A}^I$ is the set of all functions from set I to $\mathbb{A}$

- endowed with the preorder

$$\varphi \vdash_I \psi \quad \text{iff} \quad \exists a \in \Phi \forall i \in I \;\; a \bullet \varphi_i \leq \psi_i$$

- for $u : J \to I$ the *reindexing map* $[u, \mathbb{A}]_\Phi = u^* : \mathbb{A}^I \to \mathbb{A}^J$ send φ to $u^*\varphi = (\varphi_{u(j)})_{j \in J}$.

Krivine Tripos (1)

In case $\mathbb{A}$ arises from an AKS as given by $\perp\!\!\!\perp \subseteq \Lambda \times \Pi$ and $\Phi = \{a \in \mathcal{P}_{\perp\!\!\!\perp}(\Lambda) \mid |a| \neq \emptyset\}$ the indexed preorder $[-, \mathbb{A}]_\Phi$ is a **tripos**, i.e.

- all $[I, \mathbb{A}]_\Phi$ are pre-Heyting-algebras whose structure is preserved by reindexing

- for every $u : J \to I$ in $\mathbf{Set}$ the reindexing map u^* has a left adjoint $\exists_u$ and a right adjoint $\forall_u$ satisfying the (Beck-)Chevalley condition

- there is a *generic predicate* $T \in [\Sigma, \mathbb{A}]_\Phi$, namely $\Sigma = \mathbb{A}$ and $T = \mathrm{id}_\mathbb{A}$, of which all other predicates arise by reindexing since $\varphi = \varphi^* \, \mathrm{id}_\mathbb{A}$

This tripos coincides with Krivine's Classical Realizability since we have

$$\varphi \vdash_M \psi \quad \text{iff} \quad \exists t \in \Lambda \forall i \in M \; t \in |\varphi_i \to \psi_i|$$

for all $\varphi, \psi \in [M, \mathbb{A}]_\Phi$.

Krivine Tripos (2)

Proof :

Suppose $\varphi \vdash_M \psi$. Then there exists $a \in \Phi$ such that $\forall i \in M \;\; a \bullet \varphi_i \leq \psi_i$. For all $i \in M$, $u \in |a|$ and $v \in |\varphi_i|$ we have $uv \in |a \bullet \varphi_i| \subseteq |\psi_i|$. Let $u \in |a|$. Then for all $i \in M$ we have $u \in |\varphi_i| \to |\psi_i|$ and thus $\mathsf{E}u \in |\varphi_i \to \psi_i|$. Thus $t = \mathsf{E}u$ does the job.

Suppose there exists a $t \in \Lambda$ such that

$$\forall i \in M \; t \in |\varphi_i \to \psi_i|$$

Then we have

$$\forall i \in M \; \{t\}^{\perp\perp} \subseteq |\varphi_i \to \psi_i|$$

Thus for $a = \{t\}^{\perp} \in \Phi$ we have

$$\forall i \in M \forall u \in |a| \forall v \in |\varphi_i| \forall \pi \in \psi_i \;\; u * v.\pi \in \perp\!\!\!\perp$$

from which it follows that

$$\forall i \in M \; a \bullet \varphi_i \leq \psi_i$$

Thus $\varphi \vdash_M \psi$.

Forcing in Class. Real. (1)

Let P be a meet-semilattice. We write pq as a shorthand for $p \wedge q$.

Let C an upward closed subset of P. With every $X \subseteq P$ one associates*

$$|X| = \{p \in P \mid \forall q\, (\mathsf{C}(pq) \to X(q))\}$$

Such subsets of P are called propositions. We say

$$p \text{ forces } X \quad \text{iff} \quad p \in |X|$$

and thus

$$\begin{array}{lll} p \text{ forces } X \to Y & \text{iff} & \forall q\, (|X|(q) \to |Y|(pq)) \\ p \text{ forces } \forall i \in I.X_i & \text{iff} & \forall i \in I.\, p \text{ forces } X_i \end{array}$$

Apparently, we have

- p forces $X \to Y$ iff
  $\forall q\, (|X|(q) \to \forall r(\mathsf{C}(pqr) \to Y(r)))$ iff
  $\forall q, r\, (\mathsf{C}(pqr) \to |X|(q) \to Y(r))$ iff
  $p \in \Big|\{qr \mid |X|(q) \to Y(r)\}\Big|$
- p forces $\forall i \in I.X_i$ iff $p \in \Big|\bigcap_{i\in I} X_i\Big|$

*Traditionally, one would associate with X the set $X^\perp = \{p \in P \mid \forall q \in X\, \neg\mathsf{C}(pq)\}$. But, classically, we have $|X| = (P \setminus X)^\perp$.

Forcing in Class. Real. (2)

Actually, in most cases P is not a meet-semilattice **but** it is so "from point of view" of $\mathsf{C} \subseteq P$. I.e. we have a binary operation on P and an element $1 \in P$ such that

$$\mathsf{C}(p(qr)) \leftrightarrow \mathsf{C}((pq)r)$$
$$\mathsf{C}(pq) \leftrightarrow \mathsf{C}(qp)$$
$$\mathsf{C}(p) \leftrightarrow \mathsf{C}(pp)$$
$$\mathsf{C}(1p) \leftrightarrow \mathsf{C}(p)$$
$$\Big(\mathsf{C}(p) \leftrightarrow \mathsf{C}(q)\Big) \rightarrow \Big(\mathsf{C}(pr) \leftrightarrow \mathsf{C}(qr)\Big)$$

together with

$$\mathsf{C}(pq) \rightarrow \mathsf{C}(p)$$

expressing that C is upward closed.

On P we may define a congruence

$$p \simeq q \equiv \forall r.\ (C(rp) \leftrightarrow C(rq))$$

w.r.t. which P is a commutative idempotent monoid, i.e. a meet-semilattice, of which C is an upward closed subset.

Forcing in Class. Real. (3)

We have seen that p forces $X \to Y$ iff

$$\forall q, r\, (C(pqr) \to |X|(q) \to Y(r))$$

Thus a term t realizes p forces $X \to Y$ iff

$$\forall q, r \forall u {\in} \mathsf{C}(pqr) \forall s {\in} |X|(q) \forall \pi {\in} Y(r)\ t * u.s.\pi \in \perp\!\!\!\perp$$

Thus, one might want to define when a pair (t, p) realizes $X \to Y$. For this purpose one has to find an AKS structure whose term part is $\Lambda \times P$. For this purpose Krivine has defined application and push as follows

$$(t, p)(s, q) = (ts, pq) \quad (t, p).(s, \pi) = (t * s, pq)$$

Moreover, from $\perp\!\!\!\perp$ he defines a new $\perp\!\!\!\perp\!\!\!\perp$ as

$$(t, p) * (\pi, q) \in \perp\!\!\!\perp\!\!\!\perp \quad \text{iff} \ \forall u \in \mathsf{C}(pq)\ t * \pi^u \in \perp\!\!\!\perp$$

where π^u is obtained from π by inserting u at its bottom.

Forcing in Class. Real. (4)

Thus, we have

$(t, p) \in |X \to Y|$
  iff
$\forall (s, q) \in |X| \forall (r, \pi) \in Y \ (t, p) * (s, q).(\pi, r) \in \perp\!\!\perp\!\!\perp$
  iff
$\forall (s, q) \in |X| \forall (r, \pi) \in Y \forall u \in \mathsf{C}(pqr) \ t * s.\pi^u \in \perp\!\!\perp$

in accordance with the above explication of t realizes p forces $X \to Y$.

In order to jump back and forth between

$$t \text{ realizes } p \text{ forces } A \quad \text{and} \quad (t', p) \in |A|$$

one needs "read" and "write" constructs in the original AKS, i.e. command χ and χ' s.t.

(read)	$\chi * t.\pi^s$	$\succeq$	$t * s.\pi$
(write)	$\chi' * t.s.\pi$	$\succeq$	$t * \pi^s$

Using these one can transform t into t' and *vice versa*.

Krivine concludes from this that for **realizing forcing one needs global memory**.

Generic Set and Ideal

In forcing one usually considers the **generic set** $\mathcal{G}$ which is the predicate on P with $\mathcal{G}(p) = \{p\}^{\perp\perp}$. Equivalently one my consider its complement, the **generic ideal** $\mathcal{J}$ with $|\mathcal{J}(p)| = \{p\}^{\perp}$, i.e.

$$\mathcal{J}(p) = \{q \in P \mid p \neq q\}$$

as $q \in |\mathcal{J}(p)|$ iff $\forall r\,(\mathsf{C}(qr) \to p \neq r)$ iff $\neg\mathsf{C}(qp)$. Obviously $p \simeq q$ iff $\forall r\,(|\mathcal{J}(p)|(r) \leftrightarrow |\mathcal{J}(q)|(r))$. More generally, we can define

$$p \preceq q \equiv \forall r\,\Big(|\mathcal{J}(q)|(r) \to |\mathcal{J}(p)|(r)\Big)$$

i.e. $\forall r\,(\mathsf{C}(rp) \to \mathsf{C}(rq))$. This defines a preorder w.r.t. which P gets a meet-semilattice $\mathbb{P}$ with greatest element 1 where pq *picks* a binary infimum of p and q.

Equivalently, we may define

$$||\mathcal{J}(p)|| = \Pi \times \{p\}$$

since $(t,q) \in |\mathcal{J}(p)|$ iff $\forall \pi\,(t,q) * (\pi,p) \in \perp\!\!\!\perp$ iff $\forall u \in \mathsf{C}(qp) \forall \pi\, t * \pi^u \in \perp\!\!\!\perp$.

$\mathcal{P}(P)$ as a cBa

For $X \in \mathcal{P}(P)$ define $\mathcal{J}(X)$ such that

$$|\mathcal{J}(X)|(q) \quad \text{iff} \quad \forall p \in X \, \neg \mathsf{C}(qp)$$

i.e. $|\mathcal{J}|(X) = X^{\perp}$.

We may extend $\preceq$ to $\mathcal{P}(P)$ as follows

$$X \preceq Y \equiv \forall r \left(|\mathcal{J}(Y)|(r) \to |\mathcal{J}(X)|(r)\right)$$

Thus $X \preceq Y$ iff $Y^{\perp} \subseteq X^{\perp}$ iff $X^{\perp\perp} \subseteq Y^{\perp\perp}$.
This endows $\mathcal{P}(P)$ with the structure of a complete boolean preorder denoted by B.
Writing $\mathcal{E}$ for the classical realizability topos arising from the original AKS the classical topos arising from the new AKS is (equivalent to) the topos $\mathsf{Sh}_{\mathcal{E}}(B)$.

NB
B is not an assembly in $\mathsf{Sh}(\mathcal{E})$ as it is uniform.
Thus the construction of $\mathsf{Sh}_{\mathcal{E}}(B)$ from $\mathcal{E}$ is **not** induced by an opca morphism.