Partial combinatory algebras (pcas) can be thought of as algebraic structures that capture a more general notion of computability than Turing machines and oracle machines.

Definition

A *partial combinatory algebra* (or pca) is a structure $\mathcal{A} = \langle A, ., \mathbf{k}, \mathbf{s} \rangle$ such that . is a partial binary operation on A, $\mathbf{k}, \mathbf{s} \in A$ and the following conditions are satisfied.

1. $\forall x, y\ (\mathbf{k}x) \downarrow$ and $\mathbf{k}xy = x$
2. $\forall x, y, z\ (\mathbf{s}xy) \downarrow$ and $\forall x, y, z\ \mathbf{s}xyz = xz(yz)$
3. A has at least 2 distinct elements

(Given $x_1, \ldots, x_n \in A$, we write $x_1 \ldots x_n$ to mean $(\ldots(((x_1.x_2).x_3).x_4) \ldots x_n) \ldots)$. $(xy) \downarrow$ indicates that $x.y$ is defined). . is referred to as application.

It is often useful to think of elements of $\mathcal{A}$ as partial functions $A \rightharpoonup A$ via application from the left.

Definition

A partial function $F : A \rightharpoonup A$ is *represented* by $x \in \mathcal{A}$ if $\forall y \in \mathcal{A}, F(y) = x.y$. If there is such an x, we say that F is *representable*.

Example (Kleene)

Define a partial binary operation, ., on $\mathbb{N}$ by $n.m = \Phi_n(m)$. Note that the representable partial functions are precisely the computable ones. We can clearly define computable functions to fulfil the roles of **s** and **k**. Here, **k** would accept a number n and generate a program that returns n on any input. **s** would define a program that given input x, returns a program that given input y, returns another program that given input z runs x and y as programs with input z, then applies the result of the former to the result of the latter. This defines a pca, referred to as $\mathcal{K}_1$.

Example (Kleene)

Given, the set of functions $\mathbb{N} \to \mathbb{N}$, we might define an application as follows

$$f * g(n) = \begin{cases} f(\langle n \rangle * \bar{g}(m)) - 1 & \text{if } \exists \text{ (least) } m \text{ st } f(\langle n \rangle * \bar{g}(m)) > 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Note that this does not always give a total function (it might only be partial). So we instead define an application

$$f.g = \begin{cases} f * g & f * g \text{ is total} \\ \text{undefined} & \text{otherwise} \end{cases}$$

This gives a pca, referred to as $\mathcal{K}_2$, where the representable functions are precisely the continuous ones.

Example (Scott)

We can define an application on $\mathcal{P}\omega$. First fix encodings of finite subsets of ω and pairs of elements of ω as elements of ω. We write $\langle,\rangle$ for the pairing function $\omega^2 \to \omega$, and write $n \subseteq A$ to mean that $n \in \omega$ encodes a finite subset of $A \in \mathcal{P}\omega$. We can now define application as

$$A.B = \{c | \langle b, c\rangle \in A, b \subseteq B\} \tag{1}$$

This forms a combinatory algebra known as the graph model (of the lambda calculus). As for $\mathcal{K}_2$, the representable functions are precisely the continuous ones.

Example (Scott)

Given a directed complete partial order (dcpo), D, we can define a dcpo, D_∞ containing D such that D_∞ is a combinatory algebra. We first define D_i inductively by $D_0 = D$ and D_{i+1} is the dcpo of homomorphisms $D_i \to D_i$. We then define maps $\varphi_i : D_i \to D_{i+1}$, $\psi_i : D_{i+1} \to D_i$ by $\varphi_0(d) = (\lambda x).d$, $\psi_0(f) = f(\perp)$, and $\varphi_{i+1}(d) = \varphi_i \circ d \circ \psi_i$, $\psi_{i+1}(f) = \psi_i \circ f \circ \varphi_i$. D_∞ is then the inverse limit of D_i, ψ_i. Application is then defined as

$$d.d' = \sup_i d_{i+1}(d'_i) \tag{2}$$

The lambda calculus is a theory designed to model the idea that functions should be thought of as rules describing a process of converting one value to another.

Definition

A *term* (of the lambda calculus) is a member of the class defined inductively as follows

1. any of a countable supply of free variables x_i is a term
2. if s, t are terms, then $s.t$ is a term
3. if t is a term, and x a free variable, then $(\lambda x).t$ is a term

We say a variable x is bound if it appears in a term as part of a subterm of the form $(\lambda x).t$. We say that two terms are equal is one can be obtained from the other by substituting bound variables with variables not occurring in the term (so in fact terms are equivalence classes of the definition above).

We say that a term N is obtained from a term M by β reduction if M has a subterm of the form $((\lambda x).L)K$, where x does not occur in K and $((\lambda x).L)K$ is substituted in N by the term $L[x/K]$. This generates an equivalence relation. We can see that the set of equivalence classes form a pca with **s** and **k** given by

$$\mathbf{s} = (\lambda x, y, z).xz(yz) \tag{3}$$

$$\mathbf{k} = (\lambda x, y).x \tag{4}$$

This is know as the term model of the λ-calculus.

Theorem

Let $\mathcal{A} = \langle A, \mathbf{s}, \mathbf{k} \rangle$ *be a pca. Then we can find elements* $\mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{0}, \mathbf{s}_N, \mathbf{p}_N, \mathbf{d}$*, and a subset* $N \subseteq A$ *such that the following are satisfied. We write* $\mathbf{n}$ *to mean* $\mathbf{s}_N(\mathbf{n} - \mathbf{1})$*.*

1. $\forall a, b \in \mathcal{A}$, $\mathbf{p}ab \downarrow$ *and* $\mathbf{p}_0(\mathbf{p}ab) = a$, $\mathbf{p}_1(\mathbf{p}ab) = b$
2. $\mathbf{0} \in N$*, whenever* $n \in N$*, we have* $\mathbf{s}_N n \downarrow$*, and* $\mathbf{s}_N n \in N$*, and* N *is the smallest set with this property*
3. $\forall n \in N$, $\mathbf{p}_N n \downarrow$*, and* $\mathbf{p}_N(\mathbf{s}_N n) = n$
4. $\forall n, m \in N, a, b \in \mathcal{A}$, $\mathbf{d}nmab = a$ *if* $n = m$*, and* $\mathbf{d}nmab = b$ *if* $n \neq m$

Definition

IZF (Intuitionist ZF) is the theory based on Heyting predicate logic (no excluded middle) with the following axioms

1. Extensionality
2. Separation
3. Pair set
4. Power set
5. Union
6. Infinity
7. $\in$-induction
8. Collection

Definition

Let $\mathcal{A}$ be a pca. Define $V_\alpha(\mathcal{A})$ for ordinal α recursively by

$$V_{\alpha+1}(\mathcal{A}) = \mathcal{P}(\mathcal{A} \times V_\alpha(\mathcal{A})) \tag{5}$$

$$V_\lambda(\mathcal{A}) = \bigcup_{\beta<\lambda} V_\beta(\mathcal{A}) \tag{6}$$

Let $V(\mathcal{A})$ be the class given by $\bigcup_{\alpha\in\mathsf{On}} V_\alpha(\mathcal{A})$.

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Let $V(\mathcal{A})$ be the class given by $\bigcup_{\alpha\in\mathrm{On}} V_\alpha(\mathcal{A})$.

We now assume that we have made a choice of pairing and projection elements $\mathbf{p}, \mathbf{p}_0, \mathbf{p}_1$, and natural numbers. These should satisfy the conditions given in theorem 8 but won't necessarily be those that appear in the proof. We write $(x)_0$ for $\mathbf{p}_0 x$ and $(x)_1$ for $\mathbf{p}_1 x$.

Definition (Kreisel, Myhill)

$$
\begin{array}{rcl}
a \Vdash x \in y & \text{iff} & \exists \langle (a)_0, z \rangle \in y, (a)_1 \Vdash x = z \\
a \Vdash x = y & \text{iff} & \forall \langle b, z \rangle \in x, (a)_0 b \Vdash z \in y \text{ and} \\
 & & \forall \langle b, z \rangle \in y, (a)_1 b \Vdash z \in x \\
a \Vdash \phi \vee \psi & \text{iff} & \text{either } (a)_0 = \mathbf{0} \text{ and } (a)_1 \Vdash \phi, \\
 & & \text{or } (a)_0 = \mathbf{1} \text{ and } (a)_1 \Vdash \psi \\
a \Vdash \phi \wedge \psi & \text{iff} & (a)_0 \Vdash \phi \text{ and } (a)_1 \Vdash \psi \\
a \Vdash \phi \rightarrow \psi & \text{iff} & \forall b,\ b \Vdash \phi \text{ implies that } ab \Vdash \psi \\
a \Vdash \neg\phi & \text{iff} & \forall b \in \mathcal{A},\ \neg(b \Vdash \phi) \\
a \Vdash \forall x \phi & \text{iff} & \forall x \in V(\mathcal{A}), a \Vdash \phi[a/x] \\
a \Vdash \exists x \phi & \text{iff} & \exists x \in V(\mathcal{A}), a \Vdash \phi[a/x]
\end{array}
$$

We write $V(\mathcal{A}) \models \phi$ for $\exists a\; a \Vdash \phi$.

Theorem (Soundness Theorem for HPL)

Whenever ϕ is a theorem of HPL, we have

$$V(\mathcal{A}) \models \phi$$

Theorem (Soundness Theorem for IZF)

For every axiom of IZF, ϕ, we have

$$V(\mathcal{A}) \models \phi$$

We use following definition of homomorphism of pca.

Definition

Let $\mathcal{A}, \mathcal{B}$ be a pcas. Then $\alpha : \mathcal{A} \to \mathcal{B}$ is a homomorphism if

1. $\forall a, a' \in \mathcal{A}\ \alpha(aa') \simeq \alpha(a)\alpha(a')$
2. Whenever there is $a \in \mathcal{A}$ and a term t with free variables $x_1, \ldots, x_n$ such that

$$\forall a_1, \ldots, a_n\ aa_1 \ldots a_n \simeq t[x_1, \ldots, x_n / a_1, \ldots, a_n] \tag{7}$$

then also,

$$\forall b_1, \ldots, b_n\ \alpha(a)b_1 \ldots b_n \simeq t^{\alpha}[x_1, \ldots, x_n / b_1, \ldots, b_n] \tag{8}$$

Lemma

Let $\mathcal{A}$ be a pca. Then $\alpha : \mathcal{A} \to \mathcal{A}$ is an automorphism of $\mathcal{A}$ iff α is bijection and has the property

$$\forall x, y \in \mathcal{A}\ \alpha(xy) \simeq \alpha(x)\alpha(y)$$

Example

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(Topological) automorphisms of D lift to (pca) automorphisms of D_∞.

Example

Suitable permutations of ω lift to automorphisms of $\mathcal{P}\omega$.

Example

In the term model of the lambda calculus, permutations of the free variables lift to automorphisms via substitution.

We will later be using automorphisms of the term model, $\mathcal{T}$, because of the following useful property:

Lemma

Automorphisms of $\mathcal{T}$ that transpose two free variables, x_n and $x_{n'}$ are not representable in $\mathcal{T}$.

We can show that automorphisms of $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{P}(\omega)$ are always representable.

Let $\mathcal{A}$ be a pca. Note that we can lift an automorphism, α of $\mathcal{A}$ to a permutation of $V(\mathcal{A})$ recursively by the following equation

$$\alpha(x) = \{\langle\alpha(a), \alpha(y)\rangle \mid \langle a, y\rangle \in x\} \tag{9}$$

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We would like this permutation to preserve realizability in the sense that whenever $a \Vdash \phi$, we have $\alpha(a) \Vdash \phi^\alpha$.
However, note that in the definition of $\Vdash$ we made a particular choice of pairing and projection functions. Since we have no reason to expect $\alpha(\mathbf{p}_0) = \mathbf{p}_0$, we should not expect eg that $\alpha((a)_0) = (\alpha(a))_0$. Hence we adjust the definition of $\Vdash$ so that realizability is preserved by automorphisms.

Recall that in the proof of theorem 8 we defined the pairing and projection functions as follows:

$$\mathbf{p} = (\lambda x, y)(\lambda z)zxy \tag{10}$$

$$\mathbf{p}_0 = (\lambda t)t((\lambda x, y)x) \tag{11}$$

$$\mathbf{p}_1 = (\lambda t)t((\lambda x, y)y) \tag{12}$$

Also note that true and false are often implemented in the following way:

$$F = (\lambda x, y)x \tag{13}$$

$$T = (\lambda x, y)y \tag{14}$$

We now write Pair(a) to mean $(\mathbf{p}_0 a)\downarrow$, $(\mathbf{p}_1 a)\downarrow$, and $\forall x\ ax \simeq \mathbf{p}(\mathbf{p}_0 a)(\mathbf{p}_1 a)x$.
We write False(a) for $\forall x, y\ axy \simeq x$ and True(a) for $\forall x, y\ axy \simeq x$.

Lemma

Suppose that Pair(a). *If* α *is an automorphism of* $\mathcal{A}$, *then* Pair$(\alpha(a))$ *and* $\alpha(\mathbf{p}_0 a) = \mathbf{p}_0 \alpha(a)$ *and* $\alpha(\mathbf{p}_1 a) = \mathbf{p}_1 \alpha(a)$.

Lemma

1. *If* False(a) *then* False$(\alpha(a))$
2. *If* True(a) *then* True$(\alpha(a))$
3. *There is* $d \in \mathcal{A}$ *such that for all* $x, y \in \mathcal{A}$, *and* a *such that either* False(a) *or* True(a).

$$dxya = \begin{cases} x & \text{if False}(a) \\ y & \text{if True}(a) \end{cases} \tag{15}$$

We henceforth use the following new definition of realizability

Definition

$$
\begin{aligned}
a \Vdash x \in y \quad &\text{iff} \quad \text{Pair}(a) \text{ and } \exists \langle (a)_0, z \rangle \in y, (a)_1 \Vdash x = z \\
a \Vdash x = y \quad &\text{iff} \quad \text{Pair}(a) \text{ and } \forall \langle b, z \rangle \in x, (a)_0 b \Vdash z \in y \text{ and} \\
&\qquad \forall \langle b, z \rangle \in y, (a)_1 b \Vdash z \in x \\
a \Vdash \phi \vee \psi \quad &\text{iff} \quad \text{Pair}(a) \text{ and either } \text{False}((a)_0) \text{ and } (a)_1 \Vdash \phi, \\
&\qquad \text{or } \text{True}((a)_0) \text{ and } (a)_1 \Vdash \psi \\
a \Vdash \phi \wedge \psi \quad &\text{iff} \quad \text{Pair}(a) \text{ and } (a)_0 \Vdash \phi \text{ and } (a)_1 \Vdash \psi \\
a \Vdash \phi \rightarrow \psi \quad &\text{iff} \quad \forall b, b \Vdash \phi \text{ implies that } ab \Vdash \psi \\
a \Vdash \neg\phi \quad &\text{iff} \quad \forall b \in \mathcal{A}, \ \neg(b \Vdash \phi) \\
a \Vdash \forall x \phi \quad &\text{iff} \quad \forall x \in V(\mathcal{A}), a \Vdash \phi[a/x] \\
a \Vdash \exists x \phi \quad &\text{iff} \quad \exists x \in V(\mathcal{A}), a \Vdash \phi[a/x]
\end{aligned}
$$

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Theorem (Soundness Theorem for IZF)

For every axiom of IZF, ϕ, we have

$$V(\mathcal{A}) \models \phi$$

Theorem

Let $\mathcal{A}$ be a pca, and α an automorphism of $\mathcal{A}$. Then for $a \in \mathcal{A}$, $a \Vdash \phi$ if and only if $\alpha(a) \Vdash \phi^{\alpha}$.

The method of permutation models was first developed by Fraenkel and Mostowski to show the independence of choice from ZFA. Similar ideas appear in Cohens proof using forcing to show the independence of choice, and even countable choice from ZF, by lifting permutations to automorphisms of the poset of forcing conditions. Here we show that a similar idea can be used to show again, this time using only realizability, the independence of countable choice from IZF.

Definition

Let $\mathcal{A}$ be a pca. Then a *normal filter*, Γ, is an inhabited collection of subgroups of $\mathrm{Aut}(\mathcal{A})$ closed under finite intersection, supergroups, and conjugation.

Example

Let $\mathcal{A}$ be a pca, and let Γ be the smallest normal filter containing $\mathrm{Stab}(a)$ for every $a \in \mathcal{A}$. Then if $\mathrm{Aut}(\mathcal{A})$ acts on a set, X, note that $x \in X$ has finite support precisely when $\mathrm{Stab}(x) \in \Gamma$.

Example

For any pca, $\mathcal{A}$, the set Γ containing only $\mathrm{Aut}(\mathcal{A})$ is a filter. Now if $\mathrm{Aut}(\mathcal{A})$ acts on a set, X, we can see that $\mathrm{Stab}(x) \in \Gamma$ precisely when x is invariant.

Definition

Let $\mathcal{A}$ be a pca, and Γ a normal filter over $\mathcal{A}$. Then for each ordinal, α, define $V^{\Gamma}_{\alpha}(\mathcal{A})$ as follows

$$V^{\Gamma}_{\alpha+1}(\mathcal{A}) = \mathcal{A} \times \{x \subseteq V^{\Gamma}_{\alpha} \mid \mathrm{Stab}(x) \in \Gamma\} \quad (16)$$

$$V^{\Gamma}_{\lambda}(\mathcal{A}) = \bigcup_{\beta<\lambda} V^{\Gamma}_{\beta}(\mathcal{A}) \quad (17)$$

Define $V^{\Gamma}(\mathcal{A})$ by

$$V^{\Gamma}(\mathcal{A}) = \bigcup_{\alpha \in \mathbf{On}} V^{\Gamma}_{\alpha}(\mathcal{A})$$

Lemma

Heyting predicate logic is sound with respect to $V^\Gamma(\mathcal{A})$.

Proof.

The usual proof still holds for $V^\Gamma(\mathcal{A})$ □

Lemma

$\mathrm{Aut}(\mathcal{A})$ *acts on* $V^{\Gamma}(\mathcal{A})$

Proof.

Let $x \in V^{\Gamma}(\mathcal{A})$, and $g \in \mathrm{Aut}(\mathcal{A})$. Then by $\in$-induction we can assume that for all $\langle a, y \rangle \in x$, $g(y) \in V^{\Gamma}(\mathcal{A})$. Hence it is enough to show that $\mathrm{Stab}(g(x)) \in \Gamma$. But $\mathrm{Stab}(g(x)) = g^{-1}\,\mathrm{Stab}(x) g$ and Γ is normal. □

Theorem

The axioms of IZF have realizers over $V^\Gamma(\mathcal{A})$.

We essentially follow the usual proof of the soundness theorem for $V(\mathcal{A})$, checking that any sets constructed do lie in $V^\Gamma(\mathcal{A})$.

Extensionality The usual proof still holds.

Pair Given $a, b \in V^\Gamma(\mathcal{A})$, let $c = \{\langle e, a\rangle | e \in \mathcal{A}\} \cup \{\langle e, b\rangle | e \in \mathcal{A}\}$. Note that $\text{Stab}(c) \supseteq \text{Stab}(a) \cap \text{Stab}(b)$, and hence $\text{Stab}(c) \in \Gamma$, so $c \in V^\Gamma(\mathcal{A})$. Then if e is some element of $\mathcal{A}$, and $i \Vdash \forall x(x = x)$, we can take $f = \mathbf{p}(\mathbf{p}ei)(\mathbf{p}ei)$ to get $f \Vdash a \in c \wedge b \in c$ as required.

Union Let $e = (\lambda x)\mathbf{p}xi$ and let $b = \mathrm{Un}(a)$, where for each $a \in V^{\Gamma}(\mathcal{A})$,

$$\mathrm{Un}(a) = \{\langle e, c\rangle \mid e \in \mathcal{A} \wedge \exists \langle f, x\rangle \in a \; \langle e, c\rangle \in x\}$$

Suppose $g \in \mathrm{Aut}(\mathcal{A})$ is such that $g(a) = a$. Then if $\langle e, c\rangle \in \mathrm{Un}(a)$, we know that there $\langle f, x\rangle \in a\rangle$ with $\langle e, c\rangle \in x$. Then $\langle g(f), g(x)\rangle \in a$, and $\langle g(e), g(c)\rangle \in g(x)$. Hence $\langle g(e), g(c)\rangle \in \mathrm{Un}(a)$. We deduce that $\mathrm{Stab}(\mathrm{Un}(a)) \supseteq \mathrm{Stab}(a)$, and so $\mathrm{Un}(a) \in V^{\Gamma}(\mathcal{A})$. Note also that for each a, $e \Vdash \forall b \forall c((c \in b \wedge b \in a) \rightarrow c \in \mathrm{Un}(a))$

Separation Define the operator $\mathrm{Sep}(a, \phi(x, b_1, \ldots, b_n))$, where $b_1, \ldots, b_n$ are any parameters appearing in ϕ to be

$$\{\langle e, c\rangle \mid \mathrm{Pair}(e) \wedge \langle (e)_0, c\rangle \in a \wedge (e)_1 \Vdash \phi(c, b_1, \ldots, b_n)\}$$

Note that if $g \in \mathrm{Stab}(a) \cap \mathrm{Stab}(b_1) \cap \ldots \cap \mathrm{Stab}(b_n)$, and $\langle e, c\rangle \in \mathrm{Sep}(a, \phi)$, then $\langle g((e)_0), g(c)\rangle \in a$, and $g((e)_1) \Vdash \phi(g(c), b_1, \ldots, b_n)$. As before, we get that $\mathrm{Stab}(\mathrm{Sep}(a, \phi(b_1, \ldots, b_n))) \in \Gamma$, and so $\mathrm{Sep}(a, \phi(b_1, \ldots, b_n)) \in V^\Gamma(\mathcal{A})$.

Power Set This is similar to union
Infinity Note that it is enough to find $I \in V^\Gamma(\mathcal{A})$ such that $V^\Gamma(\mathcal{A}) \Vdash 0 \in I \wedge (\forall x \in I)\,\mathrm{Succ}(x) \in I$. Define recursively $x^\mathcal{A}$ by

$$x^\mathcal{A} = \{\langle a, y^\mathcal{A}\rangle | a \in \mathcal{A}, y \in x\} \tag{18}$$

Then clearly for all x, $x^\mathcal{A} \in V^\Gamma(\mathcal{A})$. Let $I = \omega^\mathcal{A}$.
Collection Let $g \Vdash \forall x \in a \exists y \phi$. Then for $\langle h, b \in a$, there is a $c \in V^\Gamma(\mathcal{A})$ such that $gh \Vdash \phi(b, c)$. By collection (in the metatheory), we can find C such that whenever $\langle h, b\rangle \in a$ there is a $c \in C$ such that $gh \Vdash \phi(b, c)$. Note that $C' = \{\langle e, g(c)\rangle \mid e \in \mathcal{A}, g \in \mathrm{Aut}(\mathcal{A}), c \in C\}$ is invariant under automorphisms and hence an element of $V^\Gamma(\mathcal{A})$. Also, we can easily find a realizer for $V^\Gamma(\mathcal{A}) \models \forall x \in a \exists y \phi \wedge y \in C'$.
Induction The usual proof still holds.

We now show the proof that AC_ω is independent of IZF. We construct a $V^\Gamma(\mathcal{A})$ such that $V^\Gamma(\mathcal{A}) \models \neg AC_\omega$. We use $\mathcal{A} = \mathcal{T}$, the term model of the lambda calculus mentioned earlier, and Γ the "finite support" normal filter.

We construct a countable family of inhabited sets $(X_n)_{n \in \omega}$ with no choice function in $V^\Gamma(\mathcal{T})$. For $i \in \omega$, let $\underline{i} \in \mathcal{T}$ be the numeral for i. Note that since numerals contain no free variables, they are fixed by automorphisms of $\mathcal{T}$. Hence the usual $\bar{\omega}$ is an element of $V^\Gamma(\mathcal{T})$.

Now for each $n \in \omega$, let

$$\tilde{n} = \{\langle x_i, \bar{i}\rangle \mid i < n\}$$

(where x_i are the free variables in $\mathcal{T}$).
Since $\tilde{n}$ has only finitely many free variables, we can see that $\tilde{n} \in V^\Gamma(\mathcal{T})$.
Earlier we noted that automorphisms transposing free variables are not representable. This gives the following lemma:

Lemma

For each $n \in \omega$, let τ be an automorphism transposing free variables $x_i, x_{i'}$ for $i, i' < n$. Then,

$$V^\Gamma(\mathcal{T}) \not\models \tilde{n} = \tau(\tilde{n})$$

Now, for each n, let

$$X_n = \{\langle \underline{0}, \sigma(\tilde{n})\rangle \mid \sigma \in \mathrm{Aut}(\mathcal{T})\}$$

$$f = \{\langle \underline{n}, (\overline{n}, X_n) \mid n \in \omega\}$$

Note that we have defined X_n to be invariant under automorphisms, and hence we clearly have $f \in V^{\Gamma}(\mathcal{T})$.
Also we can clearly construct realizers showing that f is a function from ω to a family of inhabited sets.

Now suppose that f has a choice function, $g \in V^{\Gamma}(\mathcal{T})$. That is

$$V^{\Gamma}(\mathcal{T}) \models \forall n \in \omega \, \exists! x (n, x) \in g \tag{19}$$

$$V^{\Gamma}(\mathcal{T}) \models \forall n \in \omega \, g(n) \in f(n) \tag{20}$$

Since $g \in V^{\Gamma}(\mathcal{T})$, we know that there must exist $N \in \omega$ such that whenever $\alpha \in \mathrm{Aut}(\mathcal{T})$ fixes $x_0, \ldots x_N$, α also fixes g. Let $N' = N + 2$, and note that by 19 there must be some $x \in V^{\Gamma}(\mathcal{T})$ such that

$$V^{\Gamma}(\mathcal{T}) \models x \in X_{N'} \tag{21}$$

$$V^{\Gamma}(\mathcal{T}) \models (\overline{N'}, x) \in g \tag{22}$$

By 21 and 22, there must exist $\sigma \in \mathrm{Aut}(\mathcal{T})$, such that

$$V^{\Gamma}(\mathcal{T}) \models x = \sigma(\widetilde{N'}) \quad (23)$$

$$V^{\Gamma}(\mathcal{T}) \models (\overline{N'}, \sigma(\widetilde{N'})) \in g \quad (24)$$

Now choose n, n' such that $\sigma(n), \sigma(n') > N$, and let τ be the automorphism lifted from the transposition of $\sigma(n)$ and $\sigma(n')$. Then τ fixes g, and so by 24,

$$V^{\Gamma}(\mathcal{T}) \models (\overline{N'}, \tau \circ \sigma(\widetilde{N'})) \in g \quad (25)$$

Then applying 19 gives

$$V^{\Gamma}(\mathcal{T}) \models \tau \circ \sigma(\widetilde{N'}) = \sigma(\widetilde{N'}) \quad (26)$$

Finally, applying σ^{-1} gives

$$V^{\Gamma}(\mathcal{T}) \models \sigma^{-1} \circ \tau \circ \sigma(\widetilde{N'}) = \widetilde{N'} \tag{27}$$

and we get a contradiction by our earlier lemma.

Note that we can show that for any n,

$$V^{\Gamma}(\mathcal{T}) \models \forall x\,(x \in \overline{n} \to \neg\neg x \in \tilde{n}) \wedge (x \in \tilde{n} \to \neg\neg x \in \overline{n})$$

So, if we could switch to classical logic at this point, we would get a proof that, for each n, $\tilde{n} = \overline{n}$, and hence that each X_n is a singleton.

Hence, this proof shows that it is consistent with IZF that there is a sequence of inhabited sets with no choice function such that each set is "almost" a singleton.