Double-glueing and Orthogonality: Refining Models of Linear Logic through Realizability

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Summary

- Linear logic ($\sim$ 1986): a fruitful decomposition of logic
- Double-glueing: Hyland and Schalk (2002)
- A unified framework inspired from realizability
- Better understanding of constructions underlying LL models

Models from the book: Coherent spaces

Coherent spaces are a historical model of LL designed by Girard.

Historical definition

A coherent space is a pair $R = (|R|, \frown_R)$ where $\frown_R$ is a reflexive relation on $|R|$.

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Folklore definition

We pose $u \perp v$ whenever $|u \cap v| \leq 1$. A coherent space is a pair $R = (|R|, \mathcal{C}_R)$ where $\mathcal{C}_R \subseteq \mathfrak{P}(|R|)$, called the set of **cliques** of R is s.t. $\mathcal{C}_R = {\mathcal{C}_R}^{\perp\perp}$.

- A morphism from R to S is a clique of $R^\perp \parr S$

Models from the book: Phase semantics

Phase semantics is another historical (but this time complete) model of LL.

Phase semantics

Let $\mathcal{M}$ be a commutative monoid and $\perp\!\!\!\perp \subseteq \mathcal{M}$ a pole. We pose $x \perp y$ whenever $xy \in \perp\!\!\!\perp$. A **fact** is a subset $F \subseteq \mathcal{M}$ s.t. $F = F^{\perp\perp}$.

- A morphism from E to F is an element $x \in (EF^{\perp})^{\perp}$.

Models from the book: Finiteness spaces

Finiteness spaces are a more recent LL model, and in particular of differential LL.

Finiteness spaces

We pose $u \perp v$ whenever $u \cap v$ is finite. A finiteness space is a pair $R = (|R|, \mathcal{F}_R)$ where $\mathcal{F}_R \subseteq \mathfrak{P}(|R|)$, called the set of **finitary sets** of R, is s.t. $\mathcal{F}_R = {\mathcal{F}_R}^{\perp\perp}$

- Morphisms are relations preserving $\mathcal{F}$, anti-preserving $\mathcal{F}^\perp$

Reverse-engineering

We can detect a common pattern in the previous examples.

- The objects are two-parts:
  - an underlying structure (a set, a monoid, ...)
  - additional information (clique, facts, finitary sets)
- A notion of orthogonality over this information
  - restriction to closed sets $A = A^{\perp\perp}$
- Morphisms are underlying morphisms (a relation, an element) preserving orthogonality properties

Axiomatizing this properties permits to define the double-glueing construction.

Double-glueing: general idea

Let us consider any model. With much handwaving:

- Our new formulas will be pairs (U, X) where:
  - U is an abstract set of **proofs**
  - X is an abstract set of **counter-proofs**
  - both are living in the model

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The practical case

Let $\mathbf{C}$ be a model of (a subsystem of) LL, and $\bot \in \mathbf{C}$ a return type.

We define the glued category $\mathbb{G}(\mathbf{C})$ as follows:

- Objects are triples $A = (R, U, X)$ where
  - $R \in \mathbf{C}$
  - $U \subseteq \mathbf{C}(1, R)$ $\rightsquigarrow$ proofs of A: $u \Vdash^{p} A$
  - $X \subseteq \mathbf{C}(R, \bot)$ $\rightsquigarrow$ counter-proofs of A: $x \Vdash^{o} A$

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- Morphisms $f : \mathbb{G}(\mathbf{C})(A, B)$ are $f : \mathbf{C}(R, S)$ s.t.
  - $\forall u \Vdash^{p} A, u; f \Vdash^{p} B$ (i.e. $f(U) \subseteq V$)
  - $\forall y \Vdash^{o} B, f; y \Vdash^{o} A$ (i.e. $f^{-1}(Y) \subseteq X$)

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We could already lift the structure from $\mathbf{C}$ to $\mathbb{G}(\mathbf{C})$ but is is actually better to refine our definition now.

Skimming with orthogonality: the slack case

$\mathbb{G}(\mathbf{C})$ contains too much junk, so we add **orthogonality** conditions:

- We set a family of relations $\perp_R \subseteq \mathbf{C}(1, R) \times \mathbf{C}(R, \perp)$
- $\perp$ must be compatible with the structure of $\mathbf{C}$
  - Essentially forward stability of $\perp$ w.r.t the connectors (stay tuned!)

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Remark

$\mathbb{G}(\mathbf{C})$ is exactly $\mathbb{S}(\mathbf{C})$ where $\perp_R$ is the full relation, i.e. $u \perp x$ for any u and x. Hence any of the following results can be applied to $\mathbb{G}(\mathbf{C})$.

Examples of orthogonalities

- In any category, let $\perp\!\!\!\perp \subseteq \mathbf{C}(1, \bot)$ and pose $u \perp x$ whenever $u; x \in \perp\!\!\!\perp$
  - These are the **focussed** orthogonalities
  - The best case for compatibility properties
  - The full orthogonality is focussed: $\perp\!\!\!\perp = \mathbf{C}(1, \bot)$
- In the category $\mathbf{Rel}$ of sets and relations:
  - $\mathbf{Rel}(1, R) \cong \mathbf{Rel}(R, \bot) \cong \mathfrak{P}(R)$
  - $u \perp x$ whenever $u \cap x$ at most a singleton
  - $u \perp x$ whenever $u \cap x$ is finite

Lifting the structure: general case

- If $\mathbf{C}$ has some structure one can transport it onto $\mathbb{S}(\mathbf{C})$:

$$(R, U, X) * (S, V, Y) \equiv (R * S, W, Z)$$

- We need to define W and Z accordingly!
  - in particular $W \perp Z$

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- We need to define W and Z accordingly!
  - in particular $W \perp Z$
- the morphisms associated to $*$ may be lifted to $\mathbb{S}(\mathbf{C})$ too
  - provided some well-behavedness conditions on $\perp$
  - ... and $\mathbb{S}(\mathbf{C})$ shall inherit the structure from $\mathbf{C}$ for free!

Lifting the structure: Additives

Lifting the additives is the easy part: as in the intuitionnistic case!

$$\frac{}{\top_1 \Vdash^{p} \top} \qquad \frac{}{0_\bot \Vdash^{o} 0}$$

$$\frac{u_1 \Vdash^{p} A_1 \qquad u_2 \Vdash^{p} A_2}{\langle u_1 \mid u_2 \rangle \Vdash^{p} A_1 \,\&\, A_2} \qquad \frac{x_i \Vdash^{o} A_i}{\pi_i ; x_i \Vdash^{o} A_1 \,\&\, A_2}$$

$$\frac{u_i \Vdash^{p} A_i}{u_i ; \iota_i \Vdash^{p} A_1 \oplus A_2} \qquad \frac{x_1 \Vdash^{o} A_1 \qquad x_2 \Vdash^{o} A_2}{[x_1 \mid x_2] \Vdash^{o} A_1 \oplus A_2}$$

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- Some side conditions...
  - projections must be positive, that is: $w; \pi_i \perp x_i \Rightarrow w \perp \pi_i ; x_i$
  - dually injections must be negative: $u_i \perp \iota_i ; z \Rightarrow u_i ; \iota_i \perp z$

Lifting the structure: Multiplicatives

Multiplicatives are hybrid disjunction/conjunction: lifting is asymmetric...

$$\frac{}{\mathrm{id}_1 \Vdash^{p} 1} \qquad \frac{\mathrm{id}_1 \perp \chi}{\chi \Vdash^{o} 1}$$

$$\frac{u_1 \Vdash^{p} A_1 \qquad u_2 \Vdash^{p} A_2}{u_1 \otimes u_2 \Vdash^{p} A_1 \otimes A_2} \qquad \frac{\forall u_i \Vdash^{p} A_i,\ z[u_i] \Vdash^{o} A_j}{z \Vdash^{o} A_1 \otimes A_2}$$

$$\frac{\forall u \Vdash^{p} A,\ u; w \Vdash^{p} B \qquad \forall y \Vdash^{o} B,\ w; y \Vdash^{o} A}{\hat{w} \Vdash^{p} A \multimap B} \qquad \frac{u \Vdash^{p} A \qquad y \Vdash^{o} B}{u \cdot y \Vdash^{o} A \multimap B}$$

$$\frac{u^* \Vdash^{o} A^*}{u \Vdash^{p} A} \qquad \frac{x^* \Vdash^{p} A^*}{x \Vdash^{o} A}$$

Compatibility requirements

For multiplicatives, the requirements to preserve the structure are:

- for any $u : \mathbf{C}(1, R)$, $v : \mathbf{C}(1, S)$ and $z : \mathbf{C}(R \otimes S, \bot)$,

$$\left.\begin{array}{l} u \perp_R z[v] \\ v \perp_S z[u] \end{array}\right\} \Rightarrow u \otimes v \perp_{R \otimes S} z$$

- for any $u : \mathbf{C}(1, R)$, $y : \mathbf{C}(S, \bot)$ and $f : \mathbf{C}(R, S)$,

$$\left.\begin{array}{l} u; f \perp_S y \\ u \perp_R f; y \end{array}\right\} \Rightarrow \hat{f} \perp_{R \multimap S} u \multimap y$$

- for any $u : \mathbf{C}(1, R)$ and $x : \mathbf{C}(R, \bot)$,

$$u \perp_R x \Rightarrow \mathrm{id}_1 \perp_1 u; x$$

- for any $u : \mathbf{C}(1, R)$ and $x : \mathbf{C}(R, \bot)$,

$$u \perp_R x \Leftrightarrow x^* \perp_{R^*} u^*$$

Lifting the structure: Exponentials

- Lifting the exponential is quite problematic.
- We need a compatible transformation $\kappa_R : \mathbf{C}(1, R) \to \mathbf{C}(1, !R)$
  - Compatibility is expressed as a herd of coherence diagrams.
- There is no unicity of such a transformation...
  - yet a canonical one: $\kappa(u) = 1 \xrightarrow{m} !1 \xrightarrow{!u} !R$

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where $\varepsilon : \mathbf{C}(!R, R)$, $e : \mathbf{C}(!R, 1)$ and $d : \mathbf{C}(!R, !R \otimes !R)$.

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- Side conditions of positivity again

An Enlighting Example

- In $\mathbf{Rel}$, take $!A = \mathcal{M}_{fin}(A)$
  - free commutative comonoid
- Canonical transformation is:

$$\kappa(u) = \{\mu \in \mathcal{M}_{fin}(A) \mid |\mu| \subseteq u\}$$

- sounds familiar:
  - similar to multiset-$\mathbf{Coh}$
  - similar to $\mathbf{Fin}$

Towards tight categories

- The slack construction is not satisfactory enough:
  - Very few examples from the litterature
  - Still a lot of junk lying around
- But we did not reach our classical examples yet.
- We forgot a requirement: the **closedness** of (counter-)proofs sets by bi-orthogonality

Tight categories

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Revisiting our models

Now we can describe our three leading examples through tight categories.

- Coherent spaces is the tight category over $\mathbf{Rel}$ with $u \perp_{\mathbf{Coh}} x \equiv |u \cap x| \leq 1$
- Phase semantics on $(\mathcal{M}, \perp\!\!\!\perp)$ is the tight category over the one-object category $\mathbf{C}_{\mathcal{M}}$ with the $\perp\!\!\!\perp$-focussed orthogonality
- Finiteness spaces is the tight category over $\mathbf{Rel}$ with $u \perp_{\mathbf{Fin}} x \equiv |u \cap x| < \infty$

Additional requirements

In order to define the structure lifting onto $\mathbb{T}(\mathbf{C})$, we need to strengthen the hypotheses on $\perp$.

- $\perp$ must be **precise**, i.e. the forward stability for multiplicatives is also reverse
- the projections and injections must be **focussed**, i.e. both positive and negative

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Polarization

When the previous requirements are met, the following (dual) results hold:

- objets of the form $(R, X^\perp, X)$ in $\mathbb{S}(\mathbf{C})$ are stable under negative connectives ($\perp$, $\parr$, $\top$, $\&$)
- objets of the form $(R, U, U^\perp)$ in $\mathbb{S}(\mathbf{C})$ are stable under positive connectives (1, $\otimes$, 0, $\oplus$)

Lifting of linear structure...

From the previous lemma, one can deduce that we only need to close proofs (resp. counter-proofs) for positive (resp. negative) connectors.

So, for objects of $\mathbb{T}(\mathbf{C})$, we define the following (the others connectives are dual):

$$1 = (1, \{\mathrm{id}_1\}^{\perp\perp})$$

$$(R, U) \otimes (S, V) = (R \otimes S, (U \otimes V)^{\perp\perp})$$

$$\top = (\top, \mathbf{C}(1, \top))$$

$$(R, U) \,\&\, (S, V) = (R \,\&\, S, U \,\&\, V)$$

$$(R, U)^* = (R^*, U^\perp)$$

... nice try, but not yet

- Alas! This is not sufficient to lift the monoidal structure...
  - we only get a polycategory
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Lifting

Let $\mathbf{C}$ be a model of M(A)LL.

1. Suppose $\bot$ is precise and self-stable, and that the multiplicative canonical isomorphisms are focussed. Then $\mathbb{T}(\mathbf{C})$ inherits its multiplicative structure from $\mathbf{C}$.
2. Suppose that the canonical morphisms for the additive structure are focussed. Then $\mathbb{T}(\mathbf{C})$ inherits its additive structure from $\mathbf{C}$.

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Examples

Focalization

Whenever $\perp$ is focussed, all the previous conditions are automatic.

- For phase semantics, $\perp$ is $\perp\!\!\!\perp$-focussed. The previous construction applies flawlessly.
- For finiteness and coherent spaces, $\perp$ is also precise and self-stable; there are slight mismatches, but they can be worked out in a straight way

Lifting the exponential

- In order to lift the exponentials, self-stability is not sufficient
- We need a stronger (but cleaner) notion: **stability**
  - essentially $(U^{\perp\perp} \multimap V^{\perp})^{\perp} = (U \multimap V^{\perp})^{\perp}$
- Construction is similar to $\mathbb{S}(\mathbf{C})$ (up to closure):

$$!(R, U) = (!R, \kappa(U)^{\perp\perp})$$

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- Focussed orthogonalities are stable
  - but exponential from phase semantics is not of that kind
- $\perp_{\mathbf{Coh}}$ and $\perp_{\mathbf{Fin}}$ are not stable...
  - but it works anyway...
- Quite a mess!

Non-uniform exponentials

- The previous construction is defined pointwise:

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Non-uniform exponentials

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- but κ can also be defined on whole sets
  - non-uniform exponentials, as in games
  - close to explain phase semantics exponential
  - requirements less strict than the pointwise case (inclusion vs. equality)

$$U \subseteq V \Rightarrow \kappa(U) \subseteq \kappa(V)$$

$$\kappa(U); \varepsilon \subseteq U$$

$$\kappa(U); \delta \subseteq \kappa(\kappa(U))^{\perp\perp}$$

$$\dots$$

Functors for free

Double-glueing constructions come with a bunch of functors for free:

- Adjunctions between $\mathbb{G}(\mathbf{C})$, $\mathbb{S}(\mathbf{C})$, $\mathbb{T}(\mathbf{C})$
- More interestingly, if $\perp_1$ and $\perp_2$ are compatible enough, $\mathbb{T}_1(\mathbf{C})$ $\mathbb{T}_2(\mathbf{C})$ can lead to pseudo-inclusion functors
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  - which are structure preserving
- Example in $\mathbf{Rel}$ with $\perp_{\mathbf{Coh}} \subseteq \perp_{\mathbf{Fin}}$: Hyvernat's functor $\Phi : \mathbf{Coh} \to \mathbf{Fin}$ where:

$$\Phi(R, U) = (R, U^{\perp_{\mathbf{Coh}} \perp_{\mathbf{Fin}}})$$

- Requirements still unclear...

Conclusion

- A powerful construction
  - Instanciates many interesting models
- A bit too abstract (*usine à gaz* ?)
- Not very useful in the intuitionnistic case
- A tool to design new models from scratch
  - that capture interesting behaviours

Thank you for listening, folks.