Indexed preorders, uniform preorders, and PCAs

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Part I
What are indexed preorders, and what can they do for realizability?

Krivine realizability

- Λ terms, Π stacks, $\perp\!\!\!\perp \subset \Lambda \times \Pi$ pole, $\mathrm{PL} \subset \Lambda$ proof-like terms
- $P(\Pi)$ set of **truth values**

Define denotation $\|\varphi\| \subseteq \Pi$ of closed formula φ inductively

- $\|R(t_1, \ldots, t_n)\| = \|R\|(\|t_1\|, \ldots, \|t_n\|)$
- $\|\varphi \Rightarrow \psi\| = {}^{\perp}\|\varphi\| \cdot \|\psi\| = \{t \cdot \pi \mid t \in {}^{\perp}\|\varphi\|, \pi \in \|\psi\|\}$
- $\|\bot\| = \Pi$
- $\|\forall x{:}I\,.\,\varphi(x)\| = \bigcup_{i:I} \|\varphi(i)\|$
- $\|\forall X\,.\,\varphi(X)\| = \bigcup_{p:P(\Pi)} \|\varphi(P)\|$
- second order encoding for other connectives

- A closed formula φ is **realizable**, if there exists $t \in \mathrm{PL}$ such that $k \perp \|\varphi\|$
- Define order on truth values: $P \leq Q$ for $P, Q \subseteq \Pi$, if $P \Rightarrow Q$ is realizable, i.e., $\exists t \in \mathrm{PL}\,.\, t \perp {}^{\perp}P \cdot Q$.

- Order on truth values does not contain all model theoretically interesting information – need to consider *predicates*
- Given a set I, a **predicate** on I is a function

$$\varphi : I \to P(\Pi)$$

- Given predicates $\varphi, \psi : I \to P(\Pi)$, define $\varphi \leq \psi$ iff $\forall x{:}I\,.\,\varphi(x) \Rightarrow \psi(x)$ is realizable, i.e.,

$$\exists t \in \mathrm{PL}\ \forall i \in I\ .\ t \bot^{\bot} \varphi(i) \cdot \psi(i)$$

- Note order of quantifiers – *not* pointwise order of truth values
- *uniform* realizer for all i
- orders of predicate *do* encode all model-theoretically interesting information

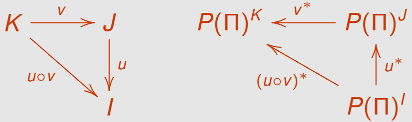
Krivine realizability as indexed boolean algebra

- For fixed set I, the set $P(\Pi)^I$ of predicates is a **boolean algebra**
- Given a function $u : J \to I$, the map

$$u^* : P(\Pi)^I \to P(\Pi)^J, \quad P(\Pi)^J \ni \varphi \mapsto \varphi \circ u$$

  is a **homomorphism** of boolean algebras.

- Given $K \xrightarrow{v} J \xrightarrow{u} I$, we have $(u \circ v)^* = v^* \circ u^*$



- We have $\mathrm{id}_I^* = \mathrm{id}_{P(\Pi)^I} : P(\Pi)^I \to P(\Pi)^I$
- The assignments

$$I \mapsto P(\Pi)^I$$
$$u \mapsto u^*$$

  define a **functor** of type $\mathbf{kt}(\perp\!\!\!\perp) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{BA}$ (**BA** is the category of boolean algebras and boolean algebra homomorphisms)

- Such a functor can be called an **indexed boolean algebra**

Quantification

- Given $u : J \to I$, define

$$\forall_u : P(\Pi)^J \to P(\Pi)^I, \qquad \forall_u(\psi)(i) = \bigcup_{uj=i} \psi(j)$$

- Then we have for $\varphi : I \to P(\Pi)$, $\psi : J \to P(\Pi)$ that

$$u^*(\varphi) \leq_J \psi \quad \text{iff} \quad \varphi \leq_I \forall_u(\psi),$$

  i.e. $u^* \dashv \forall_u$ ($\forall_u$ is right adjoint to u^*)

Generic predicate

- The map $\mathrm{tr} = \mathrm{id} : P(\Pi) \to P(\Pi)$ is a *generic predicate* for $\mathbf{kt}(\perp\!\!\!\perp)$, meaning that every predicate can be represented as a reindexing of tr.

The stated properties make $\mathbf{kt}(\perp\!\!\!\perp)$ an example of a **boolean tripos**.

Definition (Hyperdoctrine)

A **hyperdoctrine** is an indexed *Heyting algebra* $\mathcal{H} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ such that for every $u : J \to I$, $u^* : \mathcal{H}_I \to \mathcal{H}_J$ has left and right adjoints $\exists_u \dashv u^* \dashv \forall_u$, subject to the *Beck-Chevalley conditions*

$$(I \times v)^* \forall_{u \times K} \varphi \cong \forall_{u \times L} (J \times v)^* \varphi$$
$$(I \times v)^* \exists_{u \times K} \varphi \cong \exists_{u \times L} (J \times v)^* \varphi$$

hold for $u : J \to I$, $v : L \to K$.

Definition (Tripos)

A **tripos** is a hyperdoctrine $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ with a *generic family of truth values*, i.e., a predicate $\mathrm{tr} \in \mathcal{P}_{\mathrm{Prop}}$ such that for every other predicate $\varphi \in \mathcal{P}_I$ there exists $\chi_\varphi : I \to \mathrm{Prop}$ such that $\chi_\varphi^* \mathrm{tr} \cong \varphi$.

Interpreting first order logic in hyperdoctrines

Want to interpret a first order language $\mathcal{L}$ in a hyperdoctrine $\mathcal{H} : \mathbf{Set}^{op} \to \mathbf{HA}$

The language

- $\mathcal{L}$ a language of many-sorted first order logic, with sort symbols $A, B, C, \ldots$, function symbols $f, g, h, \ldots$, relation symbols $R, S, T, \ldots$
- We always consider terms $[x_1{:}A_1, \ldots, x_n{:}A_n \mid t(x_1 \ldots \times_n)]$ and formulas $[x_1{:}A_1 \ldots x_n{:}A_n \mid \varphi(x_1 \ldots x_n)]$ with *explicit variable contexts* (but we don't always write the contexts).

Interpretation of constants

- for each sort symbol A, fix a set $\|A\|$
- for each function symbol $f : A_1, \ldots, A_n \to B$, fix a function $\|f\| : \|A_1\| \times \cdots \times \|A_n\| \to \|B\|$
- For each relation symbol $R(x_1{:}A_1, \ldots, x_n{:}A_N)$ fix a relation $\|R\| \in \mathcal{H}_{\|A_1\| \times \cdots \times \|A_n\|}$

Interpretation of terms and formulas

- $\|x_1 \ldots x_n \mid x_i\| = \pi_i$ (appropriate projection)
- $\|f(t_1 \ldots t_n)\| = \|f\| \circ \langle \|t_1\| \ldots \|t_n\| \rangle$

Interpreting first order logic in hyperdoctrines (2)

Interpretation of formulas

- $\|R(t_1 \dots t_n)\| = \langle \|t_1\| \dots \|t_n\| \rangle^* \|R\|$
- $\|\varphi \wedge \psi\| = \|\varphi\| \wedge \|\psi\|$
- $\|\varphi \vee \psi\| = \|\varphi\| \vee \|\psi\|$
- $\|\varphi \Rightarrow \psi\| = \|\varphi\| \Rightarrow \|\psi\|$
- $\|\bot\| = \bot$
- $\|[x_1{:}A_1 \dots x_n{:}A_n \mid \forall y{:}B \,.\, \varphi]\| = \forall_\pi(\|[x_1{:}A_1 \dots x_n{:}A_n, y{:}B \mid \varphi]\|)$, where $\pi : \|A_1\| \times \cdots \times \|A_n\| \times \|B\| \to \|A_1\| \times \cdots \times \|A_n\|$
- analogous for $\exists$

Lemma

- *The interpretation is sound wrt intuitionistic logic, i.e.* $\|\varphi\| \cong \top$ *for intuitionistically provable* φ.
- *Interpretation in boolean hyperdoctrines is sound wrt classical logic.*

Exercise

Where do we need the Beck Chevalley condition in proving the lemma?

Recapitulating

- Indexed preorders/hyperdoctrines/triposes give a view on realizability where the concept of *predicate* is central
- Logical connectives correspond to algebraic operations characterized by universal properties (meet, join, adjunction)
- We interpret *arbitrary* formulas, not only closed ones

Define denotation $\|\varphi\| \subseteq \mathbb{N}$ of closed formula φ inductively

- $\|s = t\| = \begin{cases} \mathbb{N} & \text{if } \|s\| = \|t\| \\ \varnothing & \text{else} \end{cases}$
- $\|R(t_1, \ldots, t_n)\| = \|R\|(\|t_1\|, \ldots, \|t_n\|)$
- $\|\varphi \wedge \psi\| = \{\langle n, m\rangle \mid n \in \|\varphi\|, m \in \|\psi\|\}$
- $\|\varphi \vee \psi\| = \{\langle n, 0\rangle \mid n \in \|\varphi\|\} \cup \{\langle n, 1\rangle \mid n \in \|\psi\|\}$
- $\|\varphi \Rightarrow \psi\| = \{e \in \mathbb{N} \mid \forall n \in \|\varphi\| \,.\, \phi_e(n) \in \|\psi\|\}$
- $\|\forall x{:}I \,.\, \varphi(x)\| = \bigcap_{i \in I} \|\varphi(i)\|$ (!)
- $\|\exists x{:}I \,.\, \varphi(x)\| = \bigcup_{i \in I} \|\varphi(i)\|$ (!)

$\langle \cdot, \cdot \rangle$ (primitive) recursive pairing function, $n \mapsto \phi_n$ effective enumeration of partial recursive functions

- **Truth values** are sets $U \subseteq P(\mathbb{N})$
- **Predicates** on a set I are functions $\varphi, \psi : I \to P(\mathbb{N})$
- For predicates $\varphi, \psi : I \to P(\mathbb{N})$, define $\varphi \leq \psi$ iff $\forall x{:}I \,.\, \varphi(x) \Rightarrow \psi(x)$ is realizable, i.e.

$$\exists e{:}\mathbb{N}\ \forall i{:}I,\ n \in \varphi(i) \,.\, \phi_e(n) \in \psi(i)$$

Definition

The **effective tripos** $\mathbf{eff} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ is defined by

$$I \mapsto P(\mathbb{N})^I \quad \text{with the order defined above}$$
$$u \mapsto u^* \qquad \text{where } u^*\varphi = \varphi \circ u$$

The most important clause of Kleene's interpretation is implication, since it gives the order on predicates. The others are determined up to equivalence by soundness and universal properties.

Relativized quantification

- Kleene – doing realizability for *arithmetic* and not for generic first order logic – considered other clauses for quantification:
  - $\|\forall x{:}\mathbb{N}\,.\,\varphi(x)\|_K = \{e \mid \forall n{:}\mathbb{N}\,.\,\phi_e(n) \in \|\varphi(n)\|\}$
  - $\|\exists x{:}\mathbb{N}\,.\,\varphi(x)\|_K = \{\langle n, m\rangle \mid m \in \|\varphi(n)\|\}$
- Kleene's interpretation can be recovered up to equivalence from our *uniform* interpretation of quantifiers by **relativization**:
  - $\|\forall x{:}\mathbb{N}\,.\,\varphi(x)\|_K = \|\forall x{:}\mathbb{N}\,.\,\mathsf{nat}(x) \Rightarrow \varphi(x)\|$
  - $\|\exists x{:}\mathbb{N}\,.\,\varphi(x)\|_K = \|\exists x{:}\mathbb{N}\,.\,\mathsf{nat}(x) \wedge \varphi(x)\|$,

  where $\mathsf{nat} : \mathbb{N} \to P(\mathbb{N})$ is given by $\mathsf{nat}(n) = \{n\}$.
- This is related to Alexandre's comments about ω and $\daleth\omega$
- $\mathbb{N}$ with uniform quantification is $\daleth\omega$ – ω can be recovered by switching from the tripos to the *topos*, by a construction which formally adds subquotients to **Set** relative to the logic of the tripos
- We won't do this today, instead we talk about *partial combinatory algebras*

Partial combinatory algebras

Partial combinatory algebras provide a framework to generalize Kleene realizability.

Definition

A (weak) partial combinatory algebra (PCA) is a set $\mathcal{A}$ equipped with a partial binary operation $(- \cdot -) : \mathcal{A} \times \mathcal{A} \rightharpoonup \mathcal{A}$ such that there exist $k, s \in \mathcal{A}$ such that

- $\forall x, y \,.\, kxy = y$
- $\forall x, y \,.\, sxy{\downarrow}$
- $\forall x, y, z \,.\, xz(yz){\downarrow} \Rightarrow sxyz = xz(yz)$

Examples

- We can define a PCA structure on $\mathbb{N}$ by $n{\cdot}m = \phi_n(m)$ – existence of k and s follows from classical recursion theory
- Untyped lambda terms modulo β-equivalence form a *total* PCA with respect to application
- More generally, models of untyped lambda calculus give rise to total PCAs

PCAs via functional completeness

- Instead of using k and s, PCAs can be defined as applicative structures admitting a certain kind of abstraction operation. More precisely:

Lemma

A set $\mathcal{A}$ with a partial binary operation $(- \cdot -) : \mathcal{A} \times \mathcal{A} \rightharpoonup \mathcal{A}$ is a PCA iff for every term $t(x_1 \ldots x_{n+1})$ built up from partial application $(- \cdot -)$, parameters in $\mathcal{A}$ and variables $x_1 \ldots x_{n+1}$, there exists a term $s(x_1 \ldots x_n)$ such that

$$t(a_1 \ldots a_{n+1}) = s(a_1 \ldots a_n)\cdot a_{n+1}$$

for all $a_1 \ldots a_{n+1} \in\cdot$ whenever the left hand side is defined.

Proof.

From functional completeness, we can construct k and s by abstracting the terms $t(x, y) = x$ and $t'(x, y, z) = xz(yz)$.
Conversely, we can abstact terms using only k and s using the algorithm known from combinatory logic. □

Realizability in a PCA

Definition

For a pca $\mathcal{A}$, define the indexed preorder $\mathbf{rt}(\mathcal{A}) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ by

- Predicates on I are maps $\varphi : I \to P(\mathcal{A})$.
- For $\varphi, \psi : I \to P(\mathcal{A})$, we set

$$\varphi \leq \psi \quad \text{iff} \quad \exists e{:}\mathcal{A}\ \forall i{:}I, a \in \varphi(i) \,.\, ea \in \psi(i).$$

Lemma

$\mathbf{rt}(\mathcal{A})$ *is a tripos.*

Aim of this talk: understand and characterize indexed posets of the form $\mathbf{rt}(\mathcal{A})$ for a partial combinatory algebra $\mathcal{A}$

- Given A preorder $(A, \leq)$, we can define an indexed preorder $\langle (A, \leq) \rangle$, whose predicates are families $\varphi : I \to A$ of elements of A, and where the ordering is given pointwise (i.e. $\varphi \leq \psi : I \to A$ iff $\forall i \,.\, \varphi(i) \leq \psi(i)$).
- $\langle (A, \leq) \rangle$ is a tripos iff $(A, \leq)$ is a **complete Heyting algebra**, it is a boolean tripos iff $(A, \leq)$ is a **complete boolean algebra**
- In both cases, quantification is given by infinite meets and joins.

- Modified realizability was introduced by Kreisel, and uses terms of Gödel's system T as realizers.
- We can express it as indexed preorder as follows:
  - *truth values* are pairs (σ, S), where σ is a type of system T, and S is a set of terms of type σ modulo β-convertibility
  - *predicates* are families of truth values *of the same type*
  - For predicates φ and ψ of types σ and τ on I, define

$$\varphi \leq \psi \quad \text{iff} \quad \exists f{:}\sigma \to \tau \ \forall i{:}I, s \in \varphi(i) \, . \, fs \in \psi(i).$$

- The ensuing indexed preorder is a hyperdoctrine, but *not* a tripos (since it doesn't have a generic predicate

Part II
Uniform preorders

- PJW Hofstra, *All realizability is relative*, 2006
- J Longley, *Computability structures, simulations and realizability*, 2011
- N Hoshino, *unpublished work*, 2011

Uniform preorders

Uniform preorders are a generalization of Hofstra's *basic combinatory objects*

Definition

A **(single sorted) uniform preorder** is a pair (A, R), where A is a set, and $R \subseteq P(A \times A)$ is a set of binary relations, subject to the following axioms.

1. $r \in R,\ s \subseteq r \implies s \in R$
2. $\mathrm{id} \in R$
3. $r, s \in R \implies s \circ r \in R$

Definition

For a uniform preorder (A, R), the **associated indexed preorder** $\langle (A, R) \rangle$ has functions $\varphi : I \to A$ as predicates; the ordering relation is defined by

$$\varphi \leq \psi \quad \text{iff} \quad \{(\varphi(i), \psi(i)) \mid i{:}I\} \in R \quad \text{for} \quad \varphi, \psi : I \to A.$$

Observation

The indexed preorder $\langle (A, R) \rangle$ associated to a uniform preorder (A, R) has a generic predicate, given by $\mathrm{id}_A : A \to A$.

Examples

- For a preorder $(A, \leq)$, define a uniform preorder $(A, R_{\leq})$ by $R_{\leq} = {\downarrow}\{\leq\}$.
- Given a PCA $\mathcal{A}$, we can define a uniform preorder $(\mathcal{A}, R_{\mathcal{A}})$, where $R = \{r \subset \mathcal{A} \times \mathcal{A} \mid \exists e{:}\mathcal{A}\, \forall (a, b) \in r\,.\, e{\cdot}a = b\}$ is the set of 'sub-computable' partial functions.
- For a PCA $\mathcal{A}$, we define a *second* uniform preorder $(P\mathcal{A}, R^{\mathcal{A}})$, where $R = \{r \mid \exists e{:}\mathcal{A}\, \forall (U, V) \in r\, \forall a \in U\,.\, e{\cdot}a \in V\}$.
- Given an indexed preorder $\mathcal{D} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ with generic predicate $\iota \in \mathcal{A}_A$, we can define a uniform preorder $(A, R_{\mathcal{D}})$ by $R_{\mathcal{D}} = \{r \subset A \times A \mid \pi_l^* \iota \leq_r \pi_r^* \iota\}$, where for $r \subset A \times A$, $\pi_l, \pi_r : r \to A$ are the left and right projections.

Remarks

- For preorders $(A, \leq)$, the ordering on $\langle (A, R_{\leq}) \rangle_I$ is the pointwise one.
- For PCAs $\mathcal{A}$, we have $\langle (P\mathcal{A}, R^{\mathcal{A}}) \rangle = \mathbf{rt}(\mathcal{A})$
- For indexed preorders $\mathcal{D}$ with generic predicate $\iota \in \mathcal{D}_A$, we have $\langle (A, R_{\mathcal{D}}) \rangle \simeq \mathcal{D}$

Representability Lemma

An indexed preorder is representable by a uniform preorder iff it has a generic predicate. (Proof needs choice)

Definition

- A **monotonic map** between uniform preorders (A, R), (B, S) is a function $f : A \to B$ such that

$$r \in R \Rightarrow (f \times f)(r) \in S.$$

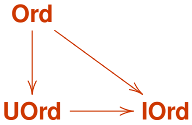
- For monotonic maps $f, g : (A, R) \to (B, S)$, we define

$$f \leq g \quad \text{iff} \quad \{(fa, ga) \mid a \in A\} \in S.$$

Remarks

- Uniform preorders and monotonous maps form an order-enriched category
- Monotonic maps between uniform preorders induce natural transformations between associated indexed preorders.

The relation between preorders, indexed preorders, and uniform preorders is displayed in the following diagram, where all inclusion functors are locally essentially full and order reflecting (i.e. they induce equivalences of preorders on hom-sets).



We can identify **Ord** with a full subcategory of **UOrd**, and **UOrd** with a full subcategory of **IOrd**.

Many sorted uniform preorders

Definition (Longley)

A **many sorted uniform preorder** is a triple (I, A, R), where $A = (A_i)_{i \in I}$ is a family of sets, and $R = (R_{ij})_{i,j \in I}$, $R_{ij} \subseteq P(A_i \times A_j)$ is a family of sets of relations, subject to the following axioms.

- $i, j \in I,\ r \in R_{ij},\ s \subseteq r \implies s \in R_{ij}$
- $i \in I \implies \mathrm{id} \in R_{ii}$
- $i, j, k \in I,\ r \in R_{ij},\ s \in R_{jk} \implies sr \in R_{ij}$

- Defined by Longley with different morphisms
- Many sorted uniform preorders correspond to indexed preorders with a generic family of predicates
- Occur e.g. in modified realizability (typed notion of realizer)
- Better closure properties, but more difficult to handle

Closure properties

- **UOrd** has small products and an involution operator $(-)^{\mathrm{op}}$
- The category of many sorted uniform preorders has moreover small coproducts and is cartesian closed $(-)^{\mathrm{op}}$

Definition (Opposite uniform preorder)

For a uniform preorder (A, R), its **opposite** $(A, R)^{\mathrm{op}}$ is given by (A, R^{op}) where $R^{\mathrm{op}} = \{r^{\circ} \mid r \in R\}$.

Definition (Binary products)

The product of uniform preorders $(A, R), (B, S)$ is given by $(A \times B, R \otimes S)$, where $R \otimes S = \downarrow\{r \times s \mid r \in R, s \in S\}$.

Finite completeness

- A preorder $(A, \leq)$ has finite meets iff $\delta : (A, \leq) \to (A, \leq) \times (A, \leq)$ and $! : (A, \leq) \to 1$ have right adjoints.
- In the same way, we say that a uniform preorder (A, R) has **finite meets** (or is **finitely complete**), if $\delta : (A, R) \to (A, R) \times (A, R)$ and $! : (A, R) \to 1$ have right adjoints.
- Since **UOrd** $\to$ **IOrd** is a local equivalence and preserves finite products, (A, R) is finitely complete iff $\langle (A, R) \rangle$ has finite meets in all fibers.
- Concretely, (A, R) has binary meets iff there exists a monotonic map $\wedge : A \times A \to A$ such that
  - $\{(a \wedge b, a) \mid a \in A, b \in A\} \in R$
  - $\{(a \wedge b, b) \mid a \in A, b \in A\} \in R$
  - $\{(a, a \wedge a) \mid a \in A\} \in R$

Examples

$(\mathcal{A}, R_{\mathcal{A}})$ and $(P\mathcal{A}, R^{\mathcal{A}})$ have finite meets for any PCA $\mathcal{A}$; $(A, R_{\leq})$ has finite meets iff $(A, \leq)$ has them.

Functional uniform preorders

Definition

We call a uniform preorder (A, R) **functional** if all elements of R are functional relations.

Lemma

If a finitely complete uniform preorder (A, R) is functional, then the pairing map $\wedge : A \times A \to A$ is injective (This is never the case for posets!).

Example

- The uniform preorder $(\mathbb{N}, \mathrm{Prim})$, where Prim is generated by the primitive recursive functions, is finitely complete and functional.
  Here, $\wedge$ is given by a primitive recursive coding of pairs.
- For any PCA $\mathcal{A}$, $(\mathcal{A}, R_{\mathcal{A}})$ is finitely complete and functional

Existential quantification

Definition

For a preorder $(A, \leq)$, denote by $\mathrm{dcl}\,(A, \leq)$ the poset of downward closed subsets of $(A, \leq)$, ordered by inclusion.

$\mathrm{dcl}\,(A, \leq)$ is a complete lattice and we have

Lemma

Given a monotonic map $f : (A, \leq) \to (B, \leq)$, where $(A, \leq)$ is a preorder, and $(B, \leq)$ is a complete lattice, there exists a unique (infinite) join-preserving map $\tilde{f} : \mathrm{dcl}\,(A, \leq) \to (B, \leq)$ making the following triangle commute.

$$\begin{array}{ccc} (A, \leq) & & \\ {\scriptstyle \downarrow\{-\}} \big\downarrow & \searrow^{f} & \\ \mathrm{dcl}\,(A, \leq) & \xrightarrow[\tilde{f}]{} & (B, \leq) \end{array}$$

Observation

Given a preorder $(A, \leq)$, we can define an ordering on PA by setting

$$M \leq N \quad \text{iff} \quad \forall m \in M\, \exists n \in N \,.\, m \leq n \quad \text{for} \quad U, V \subseteq A.$$

Then the preorder $(PA, \leq)$ is equivalent to $\mathrm{dcl}\,(A, \leq)$.

Existential quantification

For *uniform preorders*, there is a construction analogous to dcl, which in this case freely adds *existential quantification* (rather than arbitrary joins). This was originally observed by Hofstra for his BCOs.

Definition

Let (A, R) be a uniform preorder, $r \in R$. Define $[r] \subseteq P(A \times A)$ by

$$[r](M, N) :\Leftrightarrow \forall m \, \exists n \, . \, r(m, n)$$

This allows to define a uniform preorder $D(A, R) = (PA, DR)$, where $DR = \downarrow \{[r] \mid r \in R\}$.

- This gives a lax idempotent monad $D : \mathbf{UOrd} \to \mathbf{UOrd}$.
- D freely adds $\exists$ to a uniform preorder – (A, R) has $\exists$ iff it is a D-algebra
- For a pca $\mathcal{A}$, we have $D(\mathcal{A}, R_{\mathcal{A}}) = (P\mathcal{A}, R^{\mathcal{A}})$
- For preorders $(A, \leq)$, we have $D(A, R_{\leq}) \simeq (\mathrm{dcl}(A), R_{\subseteq})$
- By dualizing, we obtain a monad U classifying $\forall$

Remark

For any preorder $(A, \leq)$, dcl $(A, \leq)$ has finite meets (since it is a complete lattice). The analogous statement for uniform preorders is not true, but $D(A, R)$ has finite meets whenever (A, R) has them. In this case they are given by $M \wedge N = \{m \wedge n \mid m \in M, n \in N\}$.

The monad D_+

- Replacing the powerset P by the non-empty powerset P_+ in the definition of D, we obtain a monad D_+.

Lemma

Longley's category of computability structures is the Kleisli category of **UOrd** *(many-sorted) for the monad* D_+.

Lemma

Let $\mathcal{A}, \mathcal{B}$ *be pcas. Then an applicative morphism from* $\mathcal{A}$ *to* $\mathcal{B}$ *is the same thing as a finite meet preserving monotonous map of type*

$$(\mathcal{A}, R_{\mathcal{A}}) \to D_+(\mathcal{B}, R_{\mathcal{B}})$$

Characterizing the image of D

Definition

An element $p \in D$ of a complete lattice $(D, \leq)$ is is called *completely $\vee$-prime*, if for every family $(d_i)_{i \in I}$ of elements of D we have

$$a \leq \bigvee_i d_i \implies \exists i{:}I \,.\, a \leq d_i.$$

- A preorder $(D, \leq)$ can be recovered from its lattice $\mathrm{dcl}(D)$ of downsets by taking the completely $\vee$-prime elements.
- A complete lattice is of the form $\mathrm{dcl}(D)$ iff every element of D can be represented as a supremum of completely $\vee$-prime elements.

We can do something completely analogous for *uniform* preorders.

Characterizing the image of D

Definition

Let (A, R) be a uniform preorder with existential quantification. Call a predicate $\alpha : I \to A$ **$\exists$-prime**, if for all functions $K \xrightarrow{v} J \xrightarrow{u} I$ and predicates $\varphi : K \to A$ such that $u^*\alpha \leq \exists_v \varphi$, there exists a function $w : J \to K$ such that $vw = \mathrm{id}_J$ and $u^*\alpha \leq w^*\varphi$.

Lemma

- *The image of $\eta : \langle (A, R) \rangle \to \langle D(A, R) \rangle$ coincides up to equivalence with the $\exists$-prime predicates in $\langle D(A, R) \rangle$.*
- *A uniform preorder (B, S) is of the form $D(A, R)$ for some uniform preorder (A, R) iff it has a generic $\exists$-prime predicate, and for each predicate $\varphi : I \to B$ there exists an $\exists$-prime predicate $\alpha : J \to B$ and a function $u : J \to I$ such that $\varphi \cong \exists_u \alpha$.*

- For a preorder $(A, \leq)$ with finite meets, $\mathrm{dcl}\,(A, \leq)$ is always a *complete Heyting algebra*, which allows to interpret $\forall$ and $\Rightarrow$.
- For uniform preorders, $D(A, R)$ does not necessarily have $\Rightarrow$ and $\forall$ (counterexample: $(\mathbb{N}, \mathrm{Prim})$).
- This motivates the following definition, which gives a criterion on finitely complete uniform preorders (A, R) such that $D(A, R)$ has $\Rightarrow, \forall$.

Definition

Let (A, R) be a finitely complete uniform preorder. Call (A, R) **relationally complete**, if there exists $@ \in R$ such that for all $r \in R$ there exists $\tilde{r} \in R$ such that

$$\forall a \in A\, \exists h \in A\,.\, \tilde{r}(a, h) \wedge r(a \wedge -, -) \subseteq @(h \wedge -, -)$$

Relational completeness

Lemma

Let (A, R) be a finitely complete uniform preorder. Then the following are equivalent.

- (A, R) *is relationally complete*
- $\langle D(A, R) \rangle$ *has* $\Rightarrow$ *and* $\forall$
- $\langle D(A, R) \rangle$ *is a tripos*

- Remark: This generalizes a result of Hofstra, who characterized those BCOs A such that DA is a tripos

PCAs

Definition

Let (A, R) be a finitely complete uniform preorder. A *designated truth value* is an element $a \in A$ such that $\{(\top, a)\} \in R$

Lemma

A finitely complete uniform preorder (A, R) is (up to equivalence) of the form $(\mathcal{A}, R_{\mathcal{A}})$ if and only if it is

- *relationally complete,*
- *functional, and*
- *all elements are designated truth values*

PCAs and triposes

Definition

Let $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ be an indexed preorder. We call a predicate $\iota \in \mathcal{P}_I$ **modest**, if for every predicate $\varphi \in \mathcal{P}_J$, surjective function $e : K \to J$, and function $f : K \to I$ such that $e^*\varphi \leq f^*\iota$, there exists $g : J \to I$ such that $ge = f$ and $\varphi \leq g^*\iota$.

This looks technical, but says basically that a certain relation (expressed by the span (e, f)) is functional, and thus is related to functional uniform preorders.

Lemma

A tripos $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ is of the form $\mathbf{rt}(\mathcal{A})$ for a pca $\mathcal{A}$ iff

- *$\exists$-prime predicates in $\mathcal{P}$ are closed under finite meets,*
- *$\mathcal{P}$ is generated by $\exists$-prime predicates under existential quantification*
- *all $\exists$-prime truth values (predicates in $\mathcal{P}_1$) are equivalent*
- *there exists a modest $\exists$-prime predicate $\iota \in \mathcal{P}_A$ which is generic among $\exists$-prime predicates*

Appendix
The tripos-to-topos construction

- Triposes were originally introduced as intermediate step in the construction of realizability *toposes*
- The topos **Set**$[\mathcal{P}]$ associated to a tripos $\mathcal{P} : \mathbf{Set}^{op} \to \mathbf{Ord}$ can be viewed as category which internalises the logic of $\mathcal{P}$
- Formally, **Set**$[\mathcal{P}]$ is obtained from $\mathcal{P}$ by freely adding subquotients with respect to partial equivalence relations in $\mathcal{P}$ to **Set**

Definition

Let $\mathcal{P} : \mathbf{Set}^{op} \to \mathbf{Ord}$ be a tripos. The category **Set**$[\mathcal{P}]$ is defined as follows.

- *Objects* are pairs (C, ρ) where C is a set and $\rho \in \mathcal{P}_{C \times C}$ is a *partial equivalence relation*, which means that the judgments $\rho(x, y), \rho(y, z) \vdash \rho(x, z)$ and $\rho(x, y) \vdash \rho(y, x)$ hold in the logic of $\mathcal{P}$.
- A morphism from (C, ρ) to (D, σ) is a predicate $\phi \in \mathcal{P}_{C \times D}$ such that the following judgments hold in $\mathcal{P}$.

$$\begin{aligned} \phi(x, y) &\vdash \rho(x, x) \wedge \sigma(y, y) \\ \rho(x', x), \phi(x, y), \sigma(y, y') &\vdash \rho(x', y') \\ \phi(x, y), \phi(x, y') &\vdash \sigma(y, y') \\ \rho(x, x) &\vdash \exists y \,.\, \phi(x, y) \end{aligned}$$

- Logically equivalent predicates are identified as morphisms in **Set**$[\mathcal{P}]$.
- Composition is relational composition.

The category **Set**$[\mathcal{P}]$

- The construction of **Set**$[\mathcal{P}]$ out of $\mathcal{P}$ can be performed whenever $\mathcal{P}$ has conjunction and existential quantification satisfying the *Frobenius condition*

$$\varphi \wedge \exists_u \psi \leq \exists_u u^* \varphi \wedge \psi \quad \text{for} \quad \varphi \in \mathcal{P}_I, \psi \in \mathcal{P}_J, u : J \to I$$

  (automatic in presence of implication)
- In this case, it gives an *exact category*
- If $\mathcal{P}$ is a tripos, then **Set**$[\mathcal{P}]$ is a *topos*

The internal logic

- Monomorphisms in **Set**$[\mathcal{P}]$ are interesting, since they are the predicates of the **internal logic**
- Given an object (C, ρ) in **Set**$[\mathcal{P}]$, monomorphisms with codomain (C, ρ) can be identified with predicates $\varphi \in \mathcal{P}_C$ which are *compatible* with ρ in the sense that the judgments $\varphi(x) \vdash \rho(x, x)$ and $\varphi(x), \rho(x, y) \vdash \varphi(y)$ hold in $\mathcal{P}$

The diagonal functor

Definition

Let $\mathcal{P}$ be a tripos (or more generally an indexed preorder with $\wedge$ and $\exists$ satisfying Frobenius). We define a functor

$$\Delta : \mathbf{Set} \to \mathbf{Set}[\mathcal{P}]$$

by $\Delta A = (A, =)$ (the set equipped with the discrete equivalence relation) and $\Delta(f : A \to B) = [x, y \mid fx = y]$.

Remarks

- The order of subobjects of ΔA is equivalent to $\mathcal{P}_A$
- ΔA seems to be the 'topos version' of Krivine's $\beth A$