Constructing classical realizability models of Zermelo-Fraenkel set theory

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Réalisabilité à Chambéry

Plan

Plan

Why ZF_ε ?

- A similar difficulty occurs in the construction of
  - a forcing model of ZF [Cohen'63]
  - a Boolean-valued model of ZF [Scott, Solovay, Vopěnka]
  - a realizability model of IZF [Myhill-Friedman'73, McCarty'84]
  - a classical realizability model of ZF [Krivine'00]

  which is the interpretation of the axiom of extensionality :

$$\forall x\,\forall y\,[x = y \;\Leftrightarrow\; \forall z\,(z \in x \;\Leftrightarrow\; z \in y)$$

- The reason is that in these models, sets cannot be given a canonical representation $\rightsquigarrow$ need some extensional collapse

  (A similar problem occurs in CS when manipulating sets)

- Most authors solve the problem in the model, when defining the interpretation of extensional equality and membership

- Krivine proposes to address the problem in the syntax, using a non extensional presentation of ZF called ZF_ε (= assembly language for ZF)

The language of ZF_ε

Formulas $\quad \phi, \psi \quad ::= \quad x \not\varepsilon y \quad | \quad x \notin y \quad | \quad x \subseteq y \quad | \quad \top \quad | \quad \bot \quad | \quad \phi \Rightarrow \psi \quad | \quad \forall x\, \phi$

- Abbreviations :

$$
\begin{aligned}
\neg\phi &\equiv \phi \Rightarrow \bot \\
\phi \wedge \psi &\equiv \neg(\phi \Rightarrow \psi \Rightarrow \bot) \\
\phi \vee \psi &\equiv \neg\phi \Rightarrow \neg\psi \Rightarrow \bot \\
\phi \Leftrightarrow \psi &\equiv (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)
\end{aligned}
\qquad
\begin{aligned}
x \,\varepsilon\, y &\equiv \neg(x \not\varepsilon y) \\
x \in y &\equiv \neg(x \notin y) \\
x \approx y &\equiv x \subseteq y \wedge y \subseteq x
\end{aligned}
$$

$$\exists x\, \{\phi_1 \ \&\ \cdots \ \&\ \phi_n\} \quad \equiv \quad \neg\forall x\, (\phi_1 \Rightarrow \cdots \Rightarrow \phi_n \Rightarrow \bot)$$

$$
\begin{aligned}
(\forall x \,\varepsilon\, a)\, \phi &\equiv \forall x\, (x \,\varepsilon\, a \Rightarrow \phi) \\
(\forall x \in a)\, \phi &\equiv \forall x\, (x \in a \Rightarrow \phi)
\end{aligned}
\qquad
\begin{aligned}
(\exists x \,\varepsilon\, a)\, \phi &\equiv \exists x\, \{x \,\varepsilon\, a \ \&\ \phi\} \\
(\exists x \in a)\, \phi &\equiv \exists x\, \{x \in a \ \&\ \phi\}
\end{aligned}
$$

- A formula ϕ is extensional if it does not contain $\not\varepsilon$
  - Formulas $x \in y$, $x \subseteq y$, $x \approx y$ are extensional // $x \,\varepsilon\, y$ is not.
  - Extensional formulas are the formulas of ZF

The axioms of ZF_ε

Extensionality $\forall x\,\forall y\,(x \in y \;\Leftrightarrow\; (\exists z\,\varepsilon\, y)\, x \approx z)$

$\forall x\,\forall y\,(x \subseteq y \;\Leftrightarrow\; (\forall z\,\varepsilon\, x)\, z \in y)$

Foundation $\forall \vec{z}\;[\forall x\,((\forall y\,\varepsilon\, x)\phi(y,\vec{z}) \Rightarrow \phi(x,\vec{z})) \;\Rightarrow\; \forall x\,\phi(x,\vec{z})]$

Comprehension $\forall \vec{z}\;\forall a\,\exists b\,\forall x\,(x\;\varepsilon\; b \;\Leftrightarrow\; x\;\varepsilon\; a \wedge \phi(x,\vec{z}))$

Pairing $\forall a\,\forall b\,\exists c\,\{a\;\varepsilon\; c \;\&\; b\;\varepsilon\; c\}$

Union $\forall a\,\exists b\,(\forall x\,\varepsilon\, a)\,(\forall y\,\varepsilon\, x)\, y\;\varepsilon\; b$

Powerset $\forall a\,\exists b\,\forall x\,(\exists y\,\varepsilon\, b)\,\forall z\,(z\;\varepsilon\; y \;\Leftrightarrow\; z\;\varepsilon\; x \wedge z\;\varepsilon\; a)$

Collection $\forall \vec{z}\;\forall a\,\exists b\,(\forall x\,\varepsilon\, a)\,[\exists y\,\phi(x,y,\vec{z}) \;\Rightarrow\; (\exists y\,\varepsilon\, b)\,\phi(x,y,\vec{z})]$

Infinity $\forall \vec{z}\;\forall a\,\exists b\,\{a\;\varepsilon\; b \;\&$

$(\forall x\,\varepsilon\, b)\,(\exists y\,\phi(x,y,\vec{z}) \Rightarrow (\exists y\,\varepsilon\, b)\,\phi(x,y,\vec{z}))\}$

- Proofs formalized in natural deduction + Peirce's law

The extensional relations $\in$, $\subseteq$ and $\approx$ (1/2)

- Extensionality axioms define $\in$ and $\subseteq$ by mutual induction

$$\begin{array}{rcl} x' \in y & \Leftrightarrow & (\exists y' \varepsilon y)\, x' \approx y' \\ & \Leftrightarrow & (\exists y' \varepsilon y)\, \{x' \subseteq y' \ \&\ y' \subseteq x'\} \\ x \subseteq y & \Leftrightarrow & (\forall x' \varepsilon x)\, x' \in y \\ & \Leftrightarrow & (\forall x' \varepsilon x)\, (\exists y' \varepsilon y)\, \{x' \subseteq y' \ \&\ y' \subseteq x'\} \end{array}$$

- Foundation scheme expresses that ε is well-founded :

$$\forall \vec{z}\ [\forall x\, ((\forall y\, \varepsilon\, x)\phi(y, \vec{z}) \Rightarrow \phi(x, \vec{z}))\ \Rightarrow\ \forall x\, \phi(x, \vec{z})]$$

- Combining Extensionality with Foundation, we get :

Reflexivity : $\mathrm{ZF}_\varepsilon\ \vdash\ \forall x\, (x \subseteq x)$

Induction hypothesis : $\phi(x)\ \equiv\ x \subseteq x$

- Consequences :

$$\mathrm{ZF}_\varepsilon\ \vdash\ \forall x\, (x \approx x)$$
$$\mathrm{ZF}_\varepsilon\ \vdash\ \forall x\, \forall y\, (x\ \varepsilon\ y \Rightarrow x \in y)$$

The extensional relations $\in$, $\subseteq$ and $\approx$ (2/2)

- From Extensionality, we have :

$$x \subseteq y \;\Leftrightarrow\; (\forall x' \,\varepsilon\, x)\,(\exists y' \,\varepsilon\, y)\,\{x' \subseteq y' \;\&\; y' \subseteq x'\}$$

- Combined with Foundation again, we get :

Transitivity : $\quad ZF_\varepsilon \;\vdash\; \forall x\,\forall y\,\forall z\,(x \subseteq y \Rightarrow y \subseteq z \Rightarrow x \subseteq z)$

Induction hypothesis : $\quad \phi(x) \;\equiv\; \forall y\,\forall z\,(x \subseteq y \Rightarrow y \subseteq z \Rightarrow x \subseteq z) \;\wedge\; \forall y\,\forall z\,(z \subseteq y \Rightarrow y \subseteq x \Rightarrow z \subseteq x)$

- So that :
  - Inclusion $x \subseteq y$ is a preorder
  - Extensional equality $x \approx y$ is the associated equivalence relation

- Extensional (ZF) definitions of $\subseteq$ and $\approx$ are then derivable :

$$ZF_\varepsilon \;\vdash\; \forall x\,\forall y\,[x \subseteq y \;\Leftrightarrow\; \forall z\,(z \in x \Rightarrow z \in y)]$$

$$ZF_\varepsilon \;\vdash\; \forall x\,\forall y\,[x \approx y \;\Leftrightarrow\; \forall z\,(z \in x \Leftrightarrow z \in y)]$$

Extensional peeling

- We can now derive that $\approx$ is compatible with the two primitive extensional predicates $\notin$ and $\subseteq$:

$$\begin{array}{lcl} ZF_\varepsilon & \vdash & \forall x\,\forall y\,\forall z\,(x \approx y \Rightarrow x \notin z \Rightarrow y \notin z) \\ ZF_\varepsilon & \vdash & \forall x\,\forall y\,\forall z\,(x \approx y \Rightarrow z \notin x \Rightarrow z \notin y) \\ ZF_\varepsilon & \vdash & \forall x\,\forall y\,\forall z\,(x \approx y \Rightarrow x \subseteq z \Rightarrow y \subseteq z) \\ ZF_\varepsilon & \vdash & \forall x\,\forall y\,\forall z\,(x \approx y \Rightarrow z \subseteq x \Rightarrow z \subseteq y) \end{array}$$

Extensional peeling

For any extensional formula $\phi(x, \vec{z})$:

$$ZF_\varepsilon \;\vdash\; \forall \vec{z}\,\forall x\,\forall y\,[x \approx y \;\Rightarrow\; (\phi(x, \vec{z}) \Leftrightarrow \phi(y, \vec{z}))]$$

Proof : by structural induction on $\phi(x, \vec{z})$

- Remarks :
  - Proof structurally depends on $\phi(x, \vec{z})$ $\rightsquigarrow$ non parametric
  - Only holds when $\phi(x, \vec{z})$ is extensional. Counter-example :

$$x \approx y \;\not\Rightarrow\; (x \,\varepsilon\, z \Leftrightarrow y \,\varepsilon\, z)$$

Consequences of extensional peeling

- Extensional peeling is the tool to derive the usual extensional axioms of ZF from their intensional formulation in ZF_ε. But schemes need to be restricted to extensional formulas (as in ZF)

- In ZF_ε, (intensional) Foundation and Comprehension schemes

$$\forall\vec{z}\ [\forall x\,((\forall y\,\varepsilon\,x)\phi(y,\vec{z}) \Rightarrow \phi(x,\vec{z}))\ \Rightarrow\ \forall x\,\phi(x,\vec{z})]$$
$$\forall\vec{z}\ \forall a\,\exists b\,\forall x\,(x\ \varepsilon\ b\ \Leftrightarrow\ x\ \varepsilon\ a \land \phi(x,\vec{z}))$$

hold for any formula $\phi(x,\vec{z})$ (may contain ε)

- Combined with extensional peeling, we get

Foundation & Comprehension : ZF formulation

$$ZF_\varepsilon\ \vdash\ \forall\vec{z}\ [\forall x\,((\forall y \in x)\phi(y,\vec{z}) \Rightarrow \phi(x,\vec{z}))\ \Rightarrow\ \forall x\,\phi(x,\vec{z})]$$
$$ZF_\varepsilon\ \vdash\ \forall\vec{z}\ \forall a\,\exists b\,\forall x\,(x \in b\ \Leftrightarrow\ x \in a \land \phi(x,\vec{z}))$$

for any extensional formula $\phi(x,\vec{z})$ (cannot contain ε)

Leibniz equality and intensional peeling

- Leibniz equality is definable in ZF_ε :

$$x = y \;\equiv\; \forall z\,(x \not\varepsilon\ z \Rightarrow y \not\varepsilon\ z)$$

(Could replace $\not\varepsilon$ by ε)

- Thanks to (intensional) Comprehension, we get :

Intensional peeling

For any formula $\phi(x, \vec{z})$:

$$ZF_\varepsilon \;\vdash\; \forall\vec{z}\,\forall x\,\forall y\,[x = y \;\Rightarrow\; (\phi(x, \vec{z}) \Leftrightarrow \phi(y, \vec{z}))]$$

Proof : We only need to prove $x = y \Rightarrow (\phi(y, \vec{z}) \Rightarrow \phi(x, \vec{z}))$.
(For the converse direction : replace $\phi(x, \vec{z})$ by $\neg\phi(x, \vec{z})$.)

Assume $x = y$ and $\phi(y, \vec{z})$. From Pairing, there exists u such that $y \,\varepsilon\, u$.
From Comprehension, there exists u' such that $\forall x\,(x \,\varepsilon\, u' \Leftrightarrow x \,\varepsilon\, u \wedge \phi(x, \vec{z}))$.
By construction, we have $y \,\varepsilon\, u'$ (since $y \,\varepsilon\, u$ and $\phi(y, \vec{z})$).
Since $x = y$, we get $x \,\varepsilon\, u'$ (by contraposition). Therefore : $x \,\varepsilon\, u$ and $\phi(x, \vec{z})$.

- Remarks :
  - Proof does not structurally depend on $\phi(x, \vec{z})$ $\rightsquigarrow$ parametric
  - This property holds for any formula $\phi(x, \vec{z})$.

Strong inclusion, strong equivalence

- Let
$$x \sqsubseteq y \equiv \forall z\,(z \,\varepsilon\, x \Rightarrow z \,\varepsilon\, y)$$
$$x \sim y \equiv \forall z\,(z \,\varepsilon\, x \Leftrightarrow z \,\varepsilon\, y) \quad (\Leftrightarrow \; x \sqsubseteq y \wedge y \sqsubseteq x)$$

- Remarks :
    - $x \sqsubseteq y$ is a preorder, stronger than $x \subseteq y$
    - $x \sim y$ is the associated equivalence
    - $x \sim y$ weaker than $x = y$, stronger than $x \approx y$
      (None of the converse implications is derivable)

- Going back to Comprehension :
$$\forall \vec{z}\; \forall a\, \exists b\, \forall x\,(x \,\varepsilon\, b \Leftrightarrow x \,\varepsilon\, a \wedge \phi(x, \vec{z}))$$
    - The set $b = \{x \,\varepsilon\, a \;:\; \phi(x)\}$ is unique up to $\sim$ (and thus up to $\approx$), but not up to $=$ (Leibniz equality)

Pairing and union

- In ZF_ε, the (intensional) axioms of Pairing and Union only give upper approximations of the desired sets :

$$\forall a\,\forall b\,\exists c\,\{a \mathrel{\varepsilon} c \;\&\; b \mathrel{\varepsilon} c\}$$
$$\forall a\,\exists b\,(\forall x \mathrel{\varepsilon} a)\,(\forall y \mathrel{\varepsilon} x)\,y \mathrel{\varepsilon} b$$

- Cutting them by Comprehension, we get what we expect :

$$ZF_\varepsilon \;\vdash\; \forall a\,\forall b\,\exists c'\,\forall x\,(x \mathrel{\varepsilon} c' \;\Leftrightarrow\; x = a \lor x = b)$$
$$ZF_\varepsilon \;\vdash\; \forall a\,\exists b'\,\forall x\,(x \mathrel{\varepsilon} b' \;\Leftrightarrow\; (\exists y \mathrel{\varepsilon} a)\,x \mathrel{\varepsilon} y)$$

Note that b' and c' are unique up to strong equivalence $\sim$.

- And by extensional peeling, we get :

Pairing and Union : ZF formulation

$$ZF_\varepsilon \;\vdash\; \forall a\,\forall b\,\exists c'\,\forall x\,(x \in c' \;\Leftrightarrow\; x \approx a \lor x \approx b)$$
$$ZF_\varepsilon \;\vdash\; \forall a\,\exists b'\,\forall x\,(x \in b' \;\Leftrightarrow\; (\exists y \in a)\,x \in y)$$

Powerset

- In ZF_ε, the (intensional) Powerset axiom only gives an upper approximation of the desired set :

$$\forall a \, \exists b \, \forall x \, (\exists y \, \varepsilon \, b) \, \forall z \, (z \, \varepsilon \, y \;\Leftrightarrow\; z \, \varepsilon \, x \wedge z \, \varepsilon \, a)$$

  Intuitively : b contains a copy of all sets of the form $x \cap a$

- Cutting b with Comprehension, we get :

$$\begin{aligned} ZF_\varepsilon \;\vdash\; \forall a \, \exists b' \, \{ & (\forall x \, \varepsilon \, b') \, x \sqsubseteq a \quad \& \\ & \forall x \, (x \sqsubseteq a \;\Rightarrow\; (\exists x' \, \varepsilon \, b') \, x \sim x')\} \end{aligned}$$

  Here, b' is unique up to $\approx$, but not up to $\sim$.
  Cannot do better, since $\{x : x \sqsubseteq a\}$ is a proper class in realizability models.

- And by extensional peeling, we get :

Powerset : ZF formulation

$$ZF_\varepsilon \;\vdash\; \forall a \, \exists b' \, \forall x \, (x \in b' \;\Leftrightarrow\; x \subseteq a)$$

Collection and Infinity

- ZF_ε comes with Collection and Infinity schemes :

$$\forall \vec{z}\ \forall a\, \exists b\, (\forall x\, \varepsilon\, a)\, [\exists y\, \phi(x, y, \vec{z}) \ \Rightarrow\ (\exists y\, \varepsilon\, b)\, \phi(x, y, \vec{z})]$$

$$\forall \vec{z}\ \forall a\, \exists b\, \{a\, \varepsilon\, b\ \&\ (\forall x\, \varepsilon\, b)\, (\exists y\, \phi(x, y, \vec{z}) \Rightarrow (\exists y\, \varepsilon\, b)\, \phi(x, y, \vec{z}))\}$$

for every formula $\phi(x, y, \vec{z})$

Collection and Infinity schemes : extensional formulation

$$ZF_\varepsilon\ \vdash\ \forall \vec{z}\ \forall a\, \exists b\, (\forall x\, \varepsilon\, a)\, [\exists y\, \phi(x, y, \vec{z}) \ \Rightarrow\ (\exists y \in b)\, \phi(x, y, \vec{z})]$$

$$ZF_\varepsilon\ \vdash\ \forall \vec{z}\ \forall a\, \exists b\, \{a \in b\ \&\ (\forall x \in b)\, (\exists y\, \phi(x, y, \vec{z}) \Rightarrow (\exists y \in b)\, \phi(x, y, \vec{z}))\}$$

for every extensional formula $\phi(x, y, \vec{z})$

- In general, Collection is stronger than Replacement...
  ... but in ZF, they are equivalent due to Foundation

- Infinity scheme implies the existence of infinite sets...
  ... and it is equivalent in presence of Collection

Conservativity

- All axioms of ZF are derivable in ZF_ε :

Proposition : ZF_ε is an extension of ZF

- **Collapsing ε and $\in$:** For every formula ϕ of ZF_ε, write $\phi^\dagger$ the formula of ZF obtained by collapsing $\not\varepsilon$ to $\notin$ in ϕ.

Proposition : If $ZF_\varepsilon \vdash \phi$, then $ZF \vdash \phi^\dagger$

- Therefore, if ZF is consistent, then none of the formulas

$$\exists x\, \exists y\, (x \in y \wedge x \not\varepsilon\, y), \quad \exists x\, \exists y\, (x \approx y \wedge x \neq y), \quad \text{etc.}$$

is derivable in ZF_ε ! (But they are realized. . .)

Theorem (Conservativity)

ZF_ε is a conservative extension of ZF (and thus equiconsistent)

Proof : Assume $ZF_\varepsilon \vdash \phi$, where ϕ is extensional. Then $ZF \vdash \phi^\dagger$. But $\phi^\dagger \equiv \phi$.

Plan

The λ_c-calculus (1/2)

Syntax

Terms	t, u	$::=$	$x \mid \lambda x\,.\,t \mid tu \mid \kappa \mid \mathrm{k}_\pi$	$(\kappa \in \mathcal{K})$
Stacks	π	$::=$	$\alpha \mid t \cdot \pi$	$(\alpha \in \Pi_0,\ t$ closed$)$
Processes	p, q	$::=$	$t \star \pi$	$(t$ closed$)$

- Syntax of the language is parameterized by
  - A nonempty countable set $\mathcal{K} = \{\text{œ}; \ldots\}$ of instructions
  - A nonempty countable set $\Pi_0 = \{\alpha; \ldots\}$ of stack constants
- A term is proof-like if it contains no k_π (i.e. refers to no $\alpha \in \Pi_0$)
- Notations :

Λ	$=$	set of closed terms	
Π	$=$	set of stacks	
$\Lambda \star \Pi$	$=$	set of processes	
PL	$=$	set of closed proof-like terms	$(\subseteq \Lambda)$

- Each natural number $n \in \omega$ is encoded as $\overline{n} = \overline{s}^n\overline{0}$ $(\in \text{PL})$
  where $\overline{0} \equiv \lambda xy\,.\,x$ and $\overline{s} \equiv \lambda nxy\,.\,y\,(n\,x\,y)$

The λ_c-calculus (2/2)

- We assume that the set $\Lambda \star \Pi$ comes with a preorder $p \succ p'$ of evaluation satisfying the following rules :

Krivine Abstract Machine (KAM)

Push	$tu \star \pi$	$\succ$	$t \star u \cdot \pi$
Grab	$\lambda x\,.\,t \star u \cdot \pi$	$\succ$	$t\{x := u\} \star \pi$
Save	$\text{cc} \star u \cdot \pi$	$\succ$	$u \star \mathrm{k}_\pi \cdot \pi$
Restore	$\mathrm{k}_\pi \star u \cdot \pi'$	$\succ$	$u \star \pi$
	$\ldots$		$\ldots$

(+ reflexivity & transitivity)

  - Evaluation not defined but axiomatized. The preorder $p \succ p'$ is another parameter of the calculus, just like the sets $\mathcal{K}$ and Π_0
  - Extensible machinery : can add extra instructions and rules (We shall see examples later)

- An instance of the λ_c-calculus is defined by the triple $(\mathcal{K}, \Pi_0, \succ)$

Standard algebras

- Each classical realizability model (which is based on the λ-calculus) is parameterized by a set of processes $\perp\!\!\!\perp \subseteq \Lambda \star \Pi$ which is saturated, or closed under anti-evaluation (w.r.t. $\succ$) :

$$\text{If} \quad p \succ p' \quad \text{and} \quad p' \in \perp\!\!\!\perp, \quad \text{then} \quad p \in \perp\!\!\!\perp$$

$\rightsquigarrow$ Such a set $\perp\!\!\!\perp$ is used as the pole of the model

- We call a standard algebra any pair $\mathscr{A} \equiv ((\mathcal{K}, \Pi_0, \succ), \perp\!\!\!\perp)$ formed by
  - An instance $(\mathcal{K}, \Pi_0, \succ)$ of the λ_c-calculus
  - A saturated set $\perp\!\!\!\perp \subseteq \Lambda \star \Pi$ (i.e. the pole of the algebra $\mathscr{A}$)

- We shall first see how to build a realizability model $\mathscr{M}^{(\mathscr{A})}$ from an arbitrary standard algebra $\mathscr{A}$. But this construction more generally works when $\mathscr{A}$ is an arbitrary realizability algebra

  (We shall see the general definition later)

The ground model $\mathscr{M}$

- The whole construction is parameterized by :
  - An arbitrary model $\mathscr{M}$ of ZFC, called the ground model
  - An arbitrary standard algebra $\mathscr{A} \in \mathscr{M}$, which is taken as a point of the ground model $\mathscr{M}$
- In what follows, we call a set any point of $\mathscr{M}$
  - We shall never consider sets outside $\mathscr{M}$!
  - We write $\omega \in \mathscr{M}$ the set of natural numbers in $\mathscr{M}$. Elements of ω are called the standard natural numbers [1]
  - We consider the sets Λ, Π, $\succ$, $\perp\!\!\!\perp$ that are defined from $\mathscr{A}$ as points of the ground model $\mathscr{M}$
  - All set-theoretic notations (e.g. $\mathfrak{P}(X)$, $\{x : \phi(x)\}$, etc.) are taken relatively to the ground model $\mathscr{M}$
- Only formulas (of ZF_ε) live outside the ground model $\mathscr{M}$

1. This is just a convention of terminology. The set ω might contain numbers that are non standard according to the external/ambient/intuitive/meta theory.

Building the model $\mathscr{M}^{(\mathscr{A})}$ of $\mathscr{A}$-names

- By induction on $\alpha \in On\,(\subseteq \mathscr{M})$, we define a set $\mathscr{M}_\alpha^{(\mathscr{A})}$ by

$$\mathscr{M}_\alpha \;=\; \bigcup_{\beta<\alpha} \mathfrak{P}(\mathscr{M}_\beta \times \Pi)$$

  Note that :

  - $\mathscr{M}_0^{(\mathscr{A})} \;=\; \varnothing$
  - $\mathscr{M}_{\alpha+1}^{(\mathscr{A})} \;=\; \mathfrak{P}(\mathscr{M}_\alpha^{(\mathscr{A})} \times \Pi)$
  - $\mathscr{M}_\alpha^{(\mathscr{A})} \;=\; \bigcup_{\beta<\alpha} \mathscr{M}_\beta^{(\mathscr{A})}$ (for α limit ordinal)

- We write $\mathscr{M}^{(\mathscr{A})} \;=\; \bigcup_{\alpha} \mathscr{M}_\alpha^{(\mathscr{A})}$ the (proper) class of $\mathscr{A}$-names

- Given a name $a \in \mathscr{M}^{(\mathscr{A})}$, we write
  - $\mathrm{dom}(a) = \{b \;:\; (\exists \pi \in \Pi)\ (b, \pi) \in a\}$ (the domain of a)
  - $\mathrm{rk}(a)$ the smallest $\alpha \in On$ s.t. $a \in \mathscr{M}_\alpha^{(\mathscr{A})}$ (the rank of a)

Structure of the interpretation

- Variables $x_1, \ldots, x_n, \ldots$ of the language of ZF_ε are interpreted as names $a_1, \ldots, a_n, \ldots \in \mathscr{M}^{(\mathscr{A})}$
  - We call a formula with parameters in $\mathscr{M}^{(\mathscr{A})}$ any formula of ZF_ε enriched with constants taken in $\mathscr{M}^{(\mathscr{A})}$:

$$\phi(x_1, \ldots, x_k) \ + \ a_1, \ldots, a_k \in \mathscr{M}^{(\mathscr{A})} \quad \rightsquigarrow \quad \phi(a_1, \ldots, a_k)$$

  - Formulas with parameters in $\mathscr{M}^{(\mathscr{A})}$ constitute the language of the realizability model $\mathscr{M}^{(\mathscr{A})}$

- Closed formulas ϕ with parameters in $\mathscr{M}^{(\mathscr{A})}$ are interpreted as two sets (i.e. points of $\mathscr{M}$) :
  - A falsity value $\|\phi\| \in \mathfrak{P}(\Pi)$
  - A truth value $|\phi| \in \mathfrak{P}(\Lambda)$, defined by orthogonality :

$$|\phi| \ = \ \|\phi\|^{\perp\!\!\!\perp} \ = \ \{t \in \Lambda \ : \ (\forall \pi \in \|\phi\|)\,(t \star \pi \in \perp\!\!\!\perp)\}$$

Interpreting formulas

- Given a closed formula ϕ with parameters in $\mathscr{M}^{(\mathscr{A})}$:

Falsity value $\|\phi\| \in \mathfrak{P}(\Pi)$ defined by induction on the size of ϕ

$$\|a \not\varepsilon\ b\|,\ \|a \notin b\|,\ \|a \subseteq b\| \ = \ \text{(postponed)}$$

$$\|\top\| \ = \ \varnothing \qquad\qquad \|\bot\| \ = \ \Pi$$

$$\|\phi \Rightarrow \psi\| \ = \ |\phi| \cdot \|\psi\| \ = \ \{t \cdot \pi \ : \ t \in |\phi|,\ \pi \in \|\psi\|\}$$

$$\|\forall x\, \phi(x)\| \ = \ \bigcup_{a \in \mathscr{M}^{(\mathscr{A})}} \|\phi(a)\| \ = \ \{\pi \in \Pi \ : \ (\exists a \in \mathscr{M}^{(\mathscr{A})})\ \ \pi \in \|\phi(a)\|\}$$

Truth value $|\phi| \in \mathfrak{P}(\Lambda)$ defined by orthogonality

$$|\phi| \ = \ \|\phi\|^{\perp\!\!\!\perp} \ = \ \{t \in \Lambda \ : \ (\forall \pi \in \|\phi\|)\,(t \star \pi \in \perp\!\!\!\perp)\}$$

- **Notations :**

$$t \Vdash \phi \quad \equiv \quad t \in |\phi| \qquad\qquad (t \text{ realizes } \phi)$$

$$\mathscr{M}^{(\mathscr{A})} \Vdash \phi \quad \equiv \quad \theta \Vdash \phi \text{ for some } \theta \in \mathrm{PL}$$

$$\equiv \quad |\phi| \cap \mathrm{PL} \neq \varnothing \qquad\qquad (\phi \text{ is realized})$$

Anatomy of the interpretation

- **Denotation of units :**

Falsity value $\quad \|\top\| = \varnothing \qquad \|\bot\| = \Pi \qquad$ (by definition)

Truth value $\quad |\top| = \varnothing^{\perp\!\!\!\perp} = \Lambda \qquad |\bot| = \Pi^{\perp\!\!\!\perp} \qquad$ (by orthogonality)

- **Denotation of universal quantification :**

Falsity value : $\quad \|\forall x\, \phi(x)\| = \bigcup_{a \in \mathscr{M}^{(\mathscr{A})}} \|\phi(a)\| \qquad$ (by definition)

Truth value : $\quad |\forall x\, \phi(x)| = \bigcap_{a \in \mathscr{M}^{(\mathscr{A})}} |\phi(a)| \qquad$ (by orthogonality)

- **Denotation of implication :**

Falsity value : $\quad \|\phi \Rightarrow \psi\| = |\phi| \cdot \|\psi\| \qquad$ (by definition)

Truth value : $\quad |\phi \Rightarrow \psi| \subseteq |\phi| \rightarrow |\psi| \qquad$ (by orthogonality)

writing $|\phi| \rightarrow |\psi| = \{t \in \Lambda \ :\ \forall u \in |\phi| \ \ tu \in |\psi|\}$ (realizability arrow)

1. Converse inclusion does not hold in general, unless $\perp\!\!\!\perp$ closed under **Push**
2. In all cases : If $t \in |\phi| \rightarrow |\psi|$, then $\lambda x\,.\, tx \in |\phi \Rightarrow \psi|$ (η-expansion)

Adequacy

Deduction/typing rules

$$\frac{}{\Gamma \vdash x : \phi}\ {\scriptstyle (x:\phi)\in\Gamma} \qquad \frac{}{\Gamma \vdash t : \top}\ {\scriptstyle FV(t)\subseteq \mathrm{dom}(\Gamma)} \qquad \frac{\Gamma \vdash t : \bot}{\Gamma \vdash t : \phi}$$

$$\frac{\Gamma, x : \phi \vdash t : \psi}{\Gamma \vdash \lambda x\,.\,t : \phi \Rightarrow \psi} \qquad \frac{\Gamma \vdash t : \phi \Rightarrow \psi \qquad \Gamma \vdash u : \phi}{\Gamma \vdash tu : \psi}$$

$$\frac{\Gamma \vdash t : \phi}{\Gamma \vdash t : \forall x\, \phi}\ {\scriptstyle x \notin FV(\Gamma)} \qquad \frac{\Gamma \vdash t : \forall x\, \phi}{\Gamma \vdash t : \phi\{x := e\}}\ (e \text{ first-order term})$$

$$\frac{}{\Gamma \vdash \text{cc} : ((\phi \Rightarrow \psi) \Rightarrow \phi) \Rightarrow \phi}$$

Adequacy

Given :
- a derivable judgment $x_1 : \phi_1, \ldots, x_n : \phi_n \vdash t : \phi$
- a valuation ρ (in $\mathscr{M}^{(\mathscr{A})}$) closing $\phi_1, \ldots, \phi_n, \phi$
- realizers $u_1 \Vdash \phi_1[\rho]$, ..., $u_n \Vdash \phi_n[\rho]$

We have :

$$t\{x_1 := u_1; \ldots; x_n := u_n\} \Vdash \phi[\rho]$$

Interpreting intensional membership

- Interpretation of $\notin$ reminiscent from forcing in ZF [Cohen'63] and intuitionistic realizability in IZF [Myhill-Friedman'73, McCarty'84]

- In forcing / int. realizability, a name $a \in \mathscr{M}^{(C)}$ is a set of pairs (b, p) where $p \in C$ is a certificate witnessing that $b \mathrel{\varepsilon} a$:

$$(b, p) \in a \quad \text{means :} \quad \text{“}p \text{ forces/realizes } b \mathrel{\varepsilon} a\text{”}$$

hence : $$|b \mathrel{\varepsilon} a| \;=\; \{p \in C \;:\; (b, p) \in a\}$$

  - In forcing : p is a forcing condition
  - In intuitionistic realizability : p is a realizer

- But in classical realizability, we use refutations (i.e. stacks) instead :

$$(b, \pi) \in a \quad \text{means} \quad \text{“}\pi \text{ refutes } b \notin a\text{”}$$

hence : $$\|b \notin a\| \;=\; \{\pi \in \Pi \;:\; (b, \pi) \in a\}$$

  - $\pi \in \|b \notin a\|$ implies $\mathsf{k}_\pi \Vdash b \mathrel{\varepsilon} a \; (\equiv \neg b \notin a)$
  - $\|b \notin a\| = \varnothing = \|\top\|$ as soon as $b \notin \mathrm{dom}(a)$

Interpreting atomic formulas

Interpretation of $a' \notin a$, $a \subseteq b$ and $a' \notin b$ $\quad (a, a', b \in \mathscr{M}^{(\mathscr{A})})$

$$\begin{aligned} \|a' \not\varepsilon\ a\| &= \{\pi \in \Pi \ :\ (a', \pi) \in a\} \\ \|a \subseteq b\| &= \bigcup_{a' \in \mathrm{dom}(a)} |a' \notin b| \cdot \|a' \not\varepsilon\ a\| \\ \|a' \notin b\| &= \bigcup_{b' \in \mathrm{dom}(b)} |a' \subseteq b'| \cdot |b' \subseteq a'| \cdot \|b' \not\varepsilon\ b\| \end{aligned}$$

- Def. of $\|a' \not\varepsilon\ a\|$ is primitive (i.e. non recursive)
- Def. of $\|a \subseteq b\|$ and $\|a' \notin b\|$ is mutually recursive
  - Def. of $\|a \subseteq b\|$ calls $\|a' \notin b\|$ for all $a' \in \mathrm{dom}(a)$
  - Def. of $\|a' \notin b\|$ calls $\|a' \subseteq b'\|$ and $\|b' \subseteq a'\|$ for all $b' \in \mathrm{dom}(b)$
- Hence the definition of $\|a \subseteq b\|$ for $a, b \in \mathscr{M}_\alpha^{(\mathscr{A})}$ recursively calls $\|a' \subseteq b'\|$ for $a', b' \in \mathscr{M}_\beta^{(\mathscr{A})}$ where $\beta < \alpha$

The interpretation of $\subseteq$

- Since $\|c \not\in a\| = \varnothing$ as soon as $c \notin \mathrm{dom}(a)$:

$$\begin{aligned} \|a \subseteq b\| &= \bigcup_{c \in \mathrm{dom}(a)} |c \notin b| \cdot \|c \not\in a\| \\ &= \bigcup_{c \in \mathscr{M}^{(\mathscr{A})}} |c \notin b| \cdot \|c \not\in a\| \\ &= \|\forall z\,(z \notin b \Rightarrow z \not\in a)\| \end{aligned}$$

- Hence the atomic formula $x \subseteq y$ has the very same semantics as the formula $\forall z\,(z \notin y \Rightarrow z \not\in x)$

- By adequacy, we can build $\theta \in \mathrm{PL}$ such that (Exercise : find θ)

$$\theta \Vdash \forall x \forall y\,[\forall z\,(z \notin y \Rightarrow z \not\in x) \Leftrightarrow (\forall z\,\varepsilon\,x)\,z \in y]$$

Realizing Extensionality for $\subseteq$:

$$\theta \Vdash \forall x \forall y\,(x \subseteq y \Leftrightarrow (\forall z\,\varepsilon\,x)\,z \in y)$$

The interpretation of $\notin$

- Since $\|c \not\in b\| = \varnothing$ as soon as $c \notin \mathrm{dom}(b)$:

$$\begin{aligned}\|a \notin b\| &= \bigcup_{c\in\mathrm{dom}(b)} |a \subseteq c| \cdot |c \subseteq a| \cdot \|c \not\in b\| \\ &= \bigcup_{c\in\mathscr{M}^{(\mathscr{A})}} |a \subseteq c| \cdot |c \subseteq a| \cdot \|c \not\in b\| \\ &= \|\forall z\,(a \subseteq z \Rightarrow z \subseteq a \Rightarrow z \not\in b)\|\end{aligned}$$

- Hence the atomic formula $x \notin y$ has the very same semantics as the formula $\forall z\,(x \subseteq z \Rightarrow z \subseteq x \Rightarrow z \not\in y)$
- By adequacy, we can build $\theta' \in \mathrm{PL}$ such that (Exercise : find θ')

$$\theta' \Vdash \forall x \forall y\,[\neg\forall z\,(x \subseteq z \Rightarrow z \subseteq x \Rightarrow z \not\in y) \Leftrightarrow (\exists z\,\varepsilon\,y)\,x \approx z]$$

Realizing Extensionality for $\in$:

$$\theta' \Vdash \forall x \forall y\,(x \in y \Leftrightarrow (\exists z\,\varepsilon\,y)\,x \approx z)$$

Discriminating ε and $\in$

- Let $\quad \tilde{\varnothing} = \varnothing \quad$ and $\quad \tilde{\varnothing}' = \{\tilde{\varnothing}\} \times \|\bot \Rightarrow \bot\|$

- In the case where $\perp\!\!\!\perp \neq \varnothing$, we have :

$$\Pi^{\perp\!\!\!\perp} \neq \varnothing \quad \rightsquigarrow \quad \|\bot \Rightarrow \bot\| = \Pi^{\perp\!\!\!\perp} \cdot \Pi \neq \varnothing \quad \rightsquigarrow \quad \tilde{\varnothing} \neq \tilde{\varnothing}'$$

- But both names $\tilde{\varnothing}$ and $\tilde{\varnothing}'$ represent the empty set :

1. $\theta \Vdash \forall x\,(x \not\varepsilon\ \tilde{\varnothing})$ $\qquad (\theta \in \mathrm{PL}$ arbitrary$)$
2. $\mathbf{I} \Vdash \forall x\,(x \not\varepsilon\ \tilde{\varnothing}')$
3. Therefore : $\mathscr{M}^{(\mathscr{A})} \Vdash \tilde{\varnothing} \approx \tilde{\varnothing}'$

- Writing $\quad a = \{\tilde{\varnothing}\} \times \Pi, \quad$ we get :

1. $\mathbf{I} \Vdash \tilde{\varnothing}\ \varepsilon\ a$ and $\theta \Vdash \tilde{\varnothing}' \not\varepsilon\ a$ $\qquad (\theta \in \mathrm{PL}$ arbitrary$)$
2. Therefore : $\mathscr{M}^{(\mathscr{A})} \Vdash \tilde{\varnothing} \neq \tilde{\varnothing}'$
3. Moreover : $\mathscr{M}^{(\mathscr{A})} \Vdash \tilde{\varnothing}' \in a$ $\qquad$ (since $\mathscr{M}^{(\mathscr{A})} \Vdash \tilde{\varnothing} \approx \tilde{\varnothing}'$)

Plan

Realizing the axioms of ZF_ε

- For every axiom ϕ of ZF_ε, we want to show that :
  - There is $\theta \in PL$ such that $\theta \Vdash \phi$
  - Which we write : $\mathscr{M}^{(\mathscr{A})} \Vdash \phi$
- We have already shown that :

Realizing Extensionality

$$\mathscr{M}^{(\mathscr{A})} \Vdash \forall x \forall y \, (x \in y \Leftrightarrow (\exists z \,\varepsilon\, y)\, x \approx z)$$
$$\mathscr{M}^{(\mathscr{A})} \Vdash \forall x \forall y \, (x \subseteq y \Leftrightarrow (\forall z \,\varepsilon\, x)\, z \in y)$$

- We now need to realize the following :
  - Foundation scheme
  - Comprehension scheme
  - Pairing and Union axioms
  - Powerset axiom
  - Collection & Infinity schemes (we shall only consider Collection)

Realizing Foundation

- Consider Turing's fixpoint combinator :

$$\mathbf{Y} \equiv (\lambda yf\,.\,f\,(y\,y\,f))\,(\lambda yf\,.\,f\,(y\,y\,f))$$

- We have : $\mathbf{Y} \star t \cdot \pi \succ t \star (\mathbf{Y}\,t) \cdot \pi \qquad (t \in \Lambda,\ \pi \in \Pi)$

Proposition

For any formula $\psi(x)$ with parameters in $\mathscr{M}^{(\mathscr{A})}$, we have :

$$\mathbf{Y} \Vdash \forall x\,(\forall y\,(\psi(y) \Rightarrow y \notin x) \Rightarrow \neg\psi(x)) \Rightarrow \forall x\,\neg\psi(x)$$

Proof : We show that $\mathbf{Y} \Vdash \forall x\,(\forall y\,(\psi(y) \Rightarrow y \notin x) \Rightarrow \neg\psi(x)) \Rightarrow \neg\psi(a)$ for all $a \in \mathscr{M}^{(\mathscr{A})}$, by induction on $\mathrm{rk}(a)$.

Realizing foundation

For any formula $\phi(x, \vec{z})$, we have :

$$\mathscr{M}^{(\mathscr{A})} \Vdash \forall \vec{z}\,[\forall x\,((\forall y\,\varepsilon\,x)\,\phi(y, \vec{z}) \Rightarrow \phi(x, \vec{z})) \Rightarrow \forall x\,\phi(x, \vec{z})]$$

Realizing witnessed existential formulas

Lemma

Let $\phi(x_1, \ldots, x_n, y)$ be a formula and $\theta \in \mathrm{PL}$ such that :

$$(\forall a_1, \ldots, a_n \in \mathscr{M}^{(\mathscr{A})})\ (\exists b \in \mathscr{M}^{(\mathscr{A})})\ \ \theta \Vdash \phi(a_1, \ldots, a_n, b)$$

Then : $$\lambda z\,.\,z\,\theta \ \Vdash\ \forall x_1 \cdots \forall x_n\ \exists y\ \phi(x_1, \ldots, x_n, y)$$

- More generally :

Lemma

Given – k formulas $\phi_1(\vec{x}, y)$, ..., $\phi_k(\vec{x}, y)$ $\quad(\vec{x} \equiv x_1, \ldots, x_n)$
– k terms $\theta_1, \ldots, \theta_k \in \mathrm{PL}$

such that :

$$(\forall \vec{a} \in \mathscr{M}^{(\mathscr{A})})\ (\exists b \in \mathscr{M}^{(\mathscr{A})})\ (\theta_1 \Vdash \phi_1(\vec{a}, b)\ \wedge\ \cdots\ \wedge\ \theta_k \Vdash \phi_k(\vec{a}, b))$$

Then : $$\lambda z\,.\,z\,\theta_1 \cdots \theta_k \ \Vdash\ \forall \vec{x}\ \exists y\ \{\phi_1(\vec{x}, y)\ \&\ \cdots\ \&\ \phi_k(\vec{x}, y)\}$$

Realizing Comprehension (1/2)

- Given a name $a \in \mathscr{M}^{(\mathscr{A})}$ and a formula $\phi(x)$ (with params in $\mathscr{M}^{(\mathscr{A})}$)

  Let :
  $$b \;=\; \bigcup_{c \in \mathrm{dom}(a)} \{c\} \times \|\phi(c) \Rightarrow c \not\varepsilon\ a\|$$

- By construction, we have :
  - $\mathrm{dom}(b) \subseteq \mathrm{dom}(a)$
  - $\|c \not\varepsilon\ b\| \;=\; \|\phi(c) \Rightarrow c \not\varepsilon\ a\|$ for all $c \in \mathscr{M}^{(\mathscr{A})}$

    (Since $\|c \not\varepsilon\ b\| = \varnothing = \|\phi(c) \Rightarrow c \not\varepsilon\ a\|$ as soon as $c \notin \mathrm{dom}(a)$)

- This means that :
  - $x \not\varepsilon\ b$ has the same semantics as $\phi(x) \Rightarrow x \not\varepsilon\ a$
  - $x\ \varepsilon\ b \;\equiv\; \neg x \not\varepsilon\ b$ has the same semantics as $\neg(\phi(x) \Rightarrow x \not\varepsilon\ a)$

Realizing Comprehension (2/2)

- Let θ_1 and θ_2 be proof-like terms such that :

$$\theta_1 \ \Vdash \ \forall x\,[\neg(\phi(x) \Rightarrow x \not\varepsilon\ a) \ \Rightarrow \ x \,\varepsilon\, a \wedge \phi(x)]$$
$$\theta_2 \ \Vdash \ \forall x\,[x \,\varepsilon\, a \wedge \phi(x) \ \Rightarrow \ \neg(\phi(x) \Rightarrow x \not\varepsilon\ a)]$$

- Since $x \,\varepsilon\, b$ has the same semantics as $\neg(\phi(x) \Rightarrow x \not\varepsilon\ a)$:

$$\theta_1 \ \Vdash \ \forall x\,[x \,\varepsilon\, b \ \Rightarrow \ x \,\varepsilon\, a \wedge \phi(x)]$$
$$\theta_2 \ \Vdash \ \forall x\,[x \,\varepsilon\, a \wedge \phi(x) \ \Rightarrow \ x \,\varepsilon\, b]$$
$$\lambda u\,.\,u\,\theta_1\,\theta_2 \ \Vdash \ \forall x\,[x \,\varepsilon\, b \ \Leftrightarrow \ x \,\varepsilon\, a \wedge \phi(x)]$$

- Hence (by Lemma) :

Realizing Comprehension

For every formula $\phi(\vec{z}, x)$:

$$\lambda z\,.\,z\,(\lambda u\,.\,u\,\theta_1\,\theta_2) \ \Vdash \ \forall \vec{z}\,\forall a\,\exists b\,\forall x\,(x \,\varepsilon\, b \ \Leftrightarrow \ x \,\varepsilon\, a \wedge \phi(x, \vec{z}))$$

Realizing Pairing

- Given $a, b \in \mathscr{M}^{(\mathscr{A})}$, let

$$c \;=\; \{a; b\} \times \Pi$$

- We have $\|a \notin c\| = \|b \notin c\| = \|\bot\|$, hence :

$$\begin{array}{lll} \mathbf{I} \Vdash & a \,\varepsilon\, c & (\equiv \neg a \notin c) \\ \mathbf{I} \Vdash & b \,\varepsilon\, c & (\equiv \neg b \notin c) \end{array}$$

- Hence (by Lemma) :

Realizing Pairing

$$\lambda z\,.\,z\,\mathbf{I}\,\mathbf{I} \;\Vdash\; \forall a \forall b \exists c\,\{a \,\varepsilon\, c \;\&\; b \,\varepsilon\, c\}$$

Realizing Union

- Given $a \in \mathscr{M}^{(\mathscr{A})}$, let $\quad b = \bigcup_{a' \in \mathrm{dom}(a)} a'$

Lemma

For all $a', a'' \in \mathscr{M}^{(\mathscr{A})}$: $\quad \| a'' \notin b \Rightarrow a' \notin a \| \subseteq \| a'' \notin a' \Rightarrow a' \notin a \|$

Proof : We notice that $\| a'' \notin a' \| \subseteq \| a'' \notin b \|$ as soon as $a' \in \mathrm{dom}(a)$.

- Hence

$$\mathbf{I} \ \Vdash \ \forall x \forall y \left((y \notin x \Rightarrow x \notin a) \Rightarrow (y \notin b \Rightarrow x \notin a) \right)$$

so we can find $\theta \in \mathrm{PL}$ such that :

$$\theta \ \Vdash \ \forall x \forall y \left(x \,\varepsilon\, a \Rightarrow y \,\varepsilon\, x \Rightarrow y \,\varepsilon\, b \right)$$

- Therefore :

Realizing Union

$$\lambda z \,.\, z\, \theta \ \Vdash \ \forall a \exists b \, (\forall x \,\varepsilon\, a) \, (\forall y \,\varepsilon\, x) \, y \,\varepsilon\, b$$

Realizing Powerset

- Given $a \in \mathscr{M}^{(\mathscr{A})}$, let $\quad b \;=\; \mathfrak{P}(\mathrm{dom}(a) \times \Pi) \times \Pi$
- For every $c \in \mathscr{M}^{(\mathscr{A})}$, write :

$$c_{|a} \;=\; \bigcup_{d \in \mathrm{dom}(a)} \{d\} \times \|d \varepsilon c \Rightarrow d \notin a\|$$

- We notice that :
  1. Formula $\; z \notin c_{|a} \;$ has the same semantics as $\; z \varepsilon c \Rightarrow z \notin a$. Hence there is $\theta \in \mathrm{PL}$ such that :

$$\theta \;\Vdash\; \forall z\,(z \,\varepsilon\, c_{|a} \;\Leftrightarrow\; z \,\varepsilon\, c \wedge z \,\varepsilon\, a)$$

  2. $\mathrm{dom}(c_{|a}) \in \mathfrak{P}(\mathrm{dom}(a) \times \Pi)$, hence $\|c_{|a} \notin b\| = \|\bot\|$, and thus : $\mathsf{I} \Vdash c_{|a} \,\varepsilon\, b$
- Therefore :

Realizing Powerset

$$\lambda z\,.\,z\,(\lambda z'\,.\,z'\,\mathsf{I}\,\theta) \;\Vdash\; \forall a\,\exists b\,\forall x\,(\exists y\,\varepsilon\, b)\,\forall z\,(z \,\varepsilon\, y \;\Leftrightarrow\; z \,\varepsilon\, x \wedge z \,\varepsilon\, a)$$

Realizing Collection

- Let $\phi(x, y)$ a formula with parameters in $\mathscr{M}^{(\mathscr{A})}$ and $a \in \mathscr{M}^{(\mathscr{A})}$
- Using Collection in $\mathscr{M}$, consider a set B such that :

$$(\forall c \in \mathrm{dom}(a))\,(\forall t \in \Lambda)\,[\exists d\,(d \in \mathscr{M}^{(\mathscr{A})} \ \wedge\ t \Vdash \phi(c, d)) \ \Rightarrow \ (\exists d \in B)\,(d \in \mathscr{M}^{(\mathscr{A})} \ \wedge\ t \Vdash \phi(c, d))]$$

(Wlog, we can assume that $B \subseteq \mathscr{M}^{(\mathscr{A})}$)

- Writing $b = B \times \Pi$, we have :

Lemma

For all $c \in \mathscr{M}^{(\mathscr{A})}$: $\|\forall y\,(\phi(c, y) \Rightarrow x \notin a)\| \subseteq \|\forall y\,(\phi(c, y) \Rightarrow y \notin b)\|$

- Hence $\mathbf{I} \ \Vdash\ \forall x\,[\forall y\,(\phi(x, y) \Rightarrow y \notin b) \Rightarrow \forall y\,(\phi(x, y) \Rightarrow x \notin a)]$

  so there is $\theta \in \mathrm{PL}$ s.t. : $\theta \ \Vdash\ (\forall x\,\varepsilon\,a)\,[\exists y\,\phi(x, y) \ \Rightarrow\ (\exists y\,\varepsilon\,b)\,\phi(x, y)]$

Realizing Collection

For every formula $\phi(x, y, \vec{z})$:

$$\lambda z\,.\,z\,\theta \ \Vdash\ \forall \vec{z}\,\forall a\,\exists b\,(\forall x\,\varepsilon\,a)\,[\exists y\,\phi(x, y, \vec{z}) \ \Rightarrow\ (\exists y\,\varepsilon\,b)\,\phi(x, y, \vec{z})]$$

Plan

Adding function symbols (1/2)

- It is often convenient to enrich the language of ZF_ε with a k-ary function symbol f interpreted as a k-ary class function

$$f \ : \ \underbrace{\mathscr{M}^{(\mathscr{A})} \times \cdots \times \mathscr{M}^{(\mathscr{A})}}_{k} \to \mathscr{M}^{(\mathscr{A})}$$

- We say that f is extensional when

$$\mathscr{M}^{(\mathscr{A})} \ \Vdash \ \forall \vec{x} \, \forall \vec{y} \, (\vec{x} \approx \vec{y} \ \Rightarrow \ f(\vec{x}) \approx f(\vec{y}))$$

Beware : This is usually not the case !

- But in all cases, we have

$$\mathscr{M}^{(\mathscr{A})} \ \Vdash \ \forall \vec{x} \, \forall \vec{y} \, (\vec{x} = \vec{y} \ \Rightarrow \ f(\vec{x}) = f(\vec{y}))$$

(due to intensional peeling)

Adding function symbols (2/2)

- **Example :** Consider the successor function $s(\_)$, that is defined for all $a \in \mathscr{M}^{(\mathscr{A})}$ by

$$s(a) \;=\; \{(b, \overline{0}\cdot\pi) \;:\; (b,\pi) \in \mathrm{dom}(a)\} \cup \{(a, \overline{1}\cdot\pi) \;:\; \pi \in \Pi\}$$

Intensional/extensional characterization of s

1. $\mathscr{M}^{(\mathscr{A})} \Vdash \forall x\,\forall y\,(y \mathrel{\varepsilon} s(x) \;\Leftrightarrow\; y \mathrel{\varepsilon} x \lor y = x)$
2. $\mathscr{M}^{(\mathscr{A})} \Vdash \forall x\,\forall y\,(y \in s(x) \;\Leftrightarrow\; y \in x \lor y \approx x)$
3. The successor function s is extensional

- Actually, this function is intensionally injective :

$$\mathscr{M}^{(\mathscr{A})} \Vdash \forall x\,\forall y\,(s(x) = s(y) \;\Rightarrow\; x = y)$$

Proof : Consider a function $p(\_)$ ('predecessor') such that $p(s(a)) = a$ for all $a \in \mathscr{M}^{(\mathscr{A})}$

Constructing the set $\tilde{\omega}$ of natural numbers

- Let $\tilde{0} = \varnothing$ and $\widetilde{n+1} = s(\tilde{n})$ (for all $n \in \omega$)
- Put $\tilde{\omega} = \{(\tilde{n},\ \overline{n}\cdot\pi) : n \in \omega,\ \pi \in \Pi\}$

Intensional properties of $\tilde{\omega}$

- $\mathscr{M}^{(\mathscr{A})} \Vdash \forall y\,(y \not\varepsilon\ \tilde{0})$
- $\mathscr{M}^{(\mathscr{A})} \Vdash \forall x\,\forall y\,(y\ \varepsilon\ s(x) \Leftrightarrow y\ \varepsilon\ x \lor y = x)$
- $\mathscr{M}^{(\mathscr{A})} \Vdash \tilde{0}\ \varepsilon\ \tilde{\omega}$
- $\mathscr{M}^{(\mathscr{A})} \Vdash (\forall x\,\varepsilon\,\tilde{\omega})\, s(x)\ \varepsilon\ \tilde{\omega}$
- $\mathscr{M}^{(\mathscr{A})} \Vdash \phi(\tilde{0}) \Rightarrow (\forall x\,\varepsilon\,\tilde{\omega})\,(\phi(x) \Rightarrow \phi(s(x))) \Rightarrow (\forall x\,\varepsilon\,\tilde{\omega})\,\phi(x)$

where $\phi(x)$ is any formula with parameters in $\mathscr{M}^{(\mathscr{A})}$

- **Remark :** This implementation of ω provides a canonical intensional representation of natural numbers :

$$\mathscr{M}^{(\mathscr{A})} \Vdash (\forall x\,\varepsilon\,\tilde{\omega})\,(\forall y\,\varepsilon\,\tilde{\omega})\,(x \approx y \ \Leftrightarrow\ x = y)$$

The "type" of natural numbers : $\daleth\omega$

- Recall that : $\tilde{\omega} = \{(\tilde{p},\ \overline{p}\cdot\pi) \ :\ p \in \omega,\ \pi \in \Pi\}$
  and put : $\daleth\omega = \{(\tilde{p},\ \pi) \ :\ p \in \omega,\ \pi \in \Pi\}$
  $\daleth n = \{(\tilde{p},\ \pi) \ :\ p < n,\ \pi \in \Pi\}$
- From the definition, we have : $\mathscr{M}^{(\mathscr{A})} \Vdash \tilde{\omega} \sqsubseteq \daleth\omega$
- Distinction between (intensional) elements of $\tilde{\omega}$ and of $\daleth\omega$ is the same as between natural numbers and individuals in 2nd-order logic
- Krivine showed that in some models (such as the threads model) :
  - Inclusion $\tilde{\omega} \sqsubseteq \daleth\omega$ is strict
  - $\daleth\omega$ is (intensionally) not denumerable
  - Subsets $\daleth n \sqsubseteq \daleth\omega$ have amazing (intensional) cardinality properties
- However, the set $\daleth\omega$ is extensionally equal to $\tilde{\omega}$:

$$\mathscr{M}^{(\mathscr{A})} \Vdash \daleth\omega \approx \tilde{\omega}$$

The non extensional axiom of choice (NEAC) (1/2)

- Add an instruction quote with the rule

$$\mathrm{quote} \star t \cdot u \cdot \pi \quad \succ \quad u \star \overline{n}_t \cdot \pi$$

where n_t is the index of t according to a fixed bijection $n \mapsto t_n$ from ω to Λ

- Let $\phi(x_1, \ldots, x_k, y)$ be a formula
- Consider the $(k+1)$-ary function symbol f_ϕ interpreted by [2]
  - $f_\phi(a_1, \ldots, a_k, \tilde{n}) \;=\;$ some $b \in \mathscr{M}^{(\mathscr{A})}$ s.t. $t_n \Vdash \phi(a_1, \ldots, a_k, b)$ if there is such a name b
  - $f_\phi(a_1, \ldots, a_k, b) \;=\; \tilde{\varnothing}$ in all the other cases

Lemma

$$\lambda xy \,.\, \mathrm{quote}\, y\, (x\, y) \;\Vdash\; \forall \vec{x}\, [\forall n\, (\phi(\vec{x}, f_\phi(\vec{x}, n)) \Rightarrow n \notin \tilde{\omega}) \;\Rightarrow\; \forall y\, \neg\phi(\vec{x}, y)]$$

2. Assuming that $\mathscr{M}$ interprets the choice principle (= conservative ext. of ZFC)

The non extensional axiom of choice (NEAC) (2/2)

$$\mathscr{M}^{(\mathscr{A})} \Vdash \forall \vec{x}\, [\forall n\, (\phi(\vec{x}, f_\phi(\vec{x}, n)) \Rightarrow n \notin \tilde{\omega}) \;\Rightarrow\; \forall y\, \neg\phi(\vec{x}, y)]$$

- Taking the contrapositive, we get :

Non extensional axiom of choice (NEAC)

$$\mathscr{M}^{(\mathscr{A})} \Vdash \forall \vec{x}\, [\exists y\, \phi(\vec{x}, y) \;\Rightarrow\; (\exists n\, \varepsilon\, \tilde{\omega})\, \phi(\vec{x}, f_\phi(\vec{x}, n))]$$

- **Remarks**
  - $(f_\phi(\vec{a}, n))_{n \in \tilde{\omega}}$ is a denumerable sequence of potential witnesses of the existential formula $\exists y\, \phi(\vec{a}, y)$
  - The function f_ϕ is not extensional in general, even in the case where the formula ϕ is extensional
  - Nevertheless, NEAC is strong enough to imply the axiom of dependent choices (DC)

Alternative formulation of NEAC (1/3)

NEAC : $\mathscr{M}^{(\mathscr{A})} \Vdash \forall\vec{x}\,[\exists y\,\phi(\vec{x},y) \;\Rightarrow\; (\exists n\,\varepsilon\,\tilde{\omega})\,\phi(\vec{x},f_\phi(\vec{x},n))]$

- Consider the abbreviations :

$\psi_0(\vec{x},n) \;\equiv\; \phi(\vec{x},f_\phi(\vec{x},n))$ ("there is witness at index n")

$\psi_1(\vec{x},n) \;\equiv\; (\forall m\,\varepsilon\,\tilde{\omega})\,(\psi_0(\vec{x},m) \Rightarrow m \not\varepsilon\; n)$ ("no witness below index n")

- From the minimum principle, we get :

$$\mathscr{M}^{(\mathscr{A})} \Vdash \forall\vec{x}\,[\exists y\,\phi(\vec{x},y) \;\Rightarrow\; (\exists n\,\varepsilon\,\tilde{\omega})\,\{\psi_0(\vec{x},n)\ \&\ \psi_1(\vec{x},n)\}]$$

Idea : Introduce a k-ary function h_ϕ such that

$$h_\phi(\vec{x}) \;\approx\; f_\phi(\vec{x},n)\,,$$

where n is the smallest index s.t. $\phi(\vec{x},f_\phi(\vec{x},n))$

Alternative formulation of NEAC (2/3)

- For all $\vec{a} = a_1, \ldots, a_k \in \mathscr{M}^{(\mathscr{A})}$, let :

$$h_\phi(\vec{a}) \;=\; \bigcup_{b \in D_{\vec{a}}} \{b\} \times S_{\vec{a},b}$$

where :

$$D_{\vec{a}} \;=\; \bigcup_{n \in \omega} \mathrm{dom}(f_\phi(\vec{a}, \tilde{n}))$$

$$S_{\vec{a},b} \;=\; \|(\forall n \,\varepsilon\, \tilde{\omega})\,(\psi_0(\vec{a}, n) \Rightarrow \psi_1(\vec{a}, n) \Rightarrow b \not\varepsilon\; f_\phi(\vec{a}, n))\|$$

- By def. of $h_\phi(\vec{a})$, we have for all $b \in \mathscr{M}^{(\mathscr{A})}$:

$$\|b \not\varepsilon\; h_\phi(\vec{a})\| \;=\; \|(\forall n \,\varepsilon\, \tilde{\omega})\,(\psi_0(\vec{a}, n) \Rightarrow \psi_1(\vec{a}, n) \Rightarrow b \not\varepsilon\; f_\phi(\vec{a}, n))\|$$

- Therefore :

$$\mathscr{M}^{(\mathscr{A})} \;\Vdash\; \forall \vec{x}\; \forall z\, [z \;\varepsilon\; h_\phi(\vec{x}) \;\Leftrightarrow\; (\exists n \,\varepsilon\, \tilde{\omega})\, \{\psi_0(\vec{x}, n) \;\&\; \psi_1(\vec{x}, n) \;\&\; z \;\varepsilon\; f_\phi(\vec{x}, n)\}]$$

Alternative formulation of NEAC (3/3)

- We have shown :

$$\mathscr{M}^{(\mathscr{A})} \Vdash \forall \vec{x}\ [\exists y\, \phi(\vec{x}, y) \ \Rightarrow\ (\exists n\, \varepsilon\, \tilde{\omega})\, \{\psi_0(\vec{x}, n)\ \&\ \psi_1(\vec{x}, n)\}]$$

$$\mathscr{M}^{(\mathscr{A})} \Vdash \forall \vec{x}\ \forall z\, [z\ \varepsilon\ h_\phi(\vec{x}) \ \Leftrightarrow\ (\exists n\, \varepsilon\, \tilde{\omega})\, \{\psi_0(\vec{x}, n)\ \&\ \psi_1(\vec{x}, n)\ \&\ z\ \varepsilon\ f_\phi(\vec{x}, n)\}]$$

- Combining these results, we get :

Alternative formulation of NEAC

1. For any formula $\phi(\vec{x}, y)$:

$$\mathscr{M}^{(\mathscr{A})} \Vdash \forall \vec{x}\, [\exists y\, \phi(\vec{x}, y) \ \Rightarrow\ \exists y\, \{y \sim h_\phi(\vec{x})\ \&\ \phi(\vec{x}, y)\}]$$

2. If moreover the formula $\phi(\vec{x}, y)$ is extensional :

$$\mathscr{M}^{(\mathscr{A})} \Vdash \forall \vec{x}\, [\exists y\, \phi(\vec{x}, y) \ \Leftrightarrow\ \phi(\vec{x}, h_\phi(\vec{x}))]$$

- **Beware !** The function h_ϕ is in general non extensional, even when the formula $\phi(\vec{x}, y)$ is
- But h_ϕ can be used in Comprehension, Collection, etc.

Plan

From the λ_c-calculus to realizability algebras

- **Realizability algebras** [Krivine'10]
  - Same idea as PCAs (or OPCAs), but for classical realizability
  - Each realizability algebra $\mathscr{A}$ contains a pole $\perp\!\!\!\perp$, and defines a classical realizability model $\mathscr{M}^{(\mathscr{A})}$ of ZF_ε (from a ground model $\mathscr{M}$)

    $\rightsquigarrow$ Construction of $\mathscr{M}^{(\mathscr{A})}$ is the same as in the standard case
- Realizability algebras may be built from
  - The λ_c-calculus or Parigot's $\lambda\mu$-calculus
  - Curien-Herbelin's $\bar{\lambda}\mu$-calculus (CBN or CBV)
  - Any complete Boolean algebra
- Realizability algebras can combine (standard) classical realizability with Cohen forcing $\rightsquigarrow$ iterated forcing [Krivine'10]
- **Slogan :** classical realizability $=$ non commutative forcing

Realizability algebras (1/2) [Krivine'10]

Some terminology (where A is a fixed set) :

- Proof-term $\equiv$ λ-term with œ

  Proof-terms $t, u \quad ::= \quad x \quad | \quad \lambda x \,.\, t \quad | \quad tu \quad | \quad \text{œ}$

- A-environment $\equiv$ finite association list $\sigma \in (\mathrm{Var} \times A)^*$
  - Notations :

$$\begin{array}{rcl} \sigma & \equiv & x_1 := a_1, \ldots, x_n := a_n \\ \mathrm{dom}(\sigma) & = & \{x_1; \ldots; x_n\} \\ \mathrm{cod}(\sigma) & = & \{a_1; \ldots; a_n\} \end{array}$$

  - Environments are ordered, variables may be bound several times

- Compilation function into A $\equiv$ function $(t, \sigma) \mapsto t[\sigma]$
  - taking : proof-term t $+$ A-environment σ closing t,
  - returning : element $t[\sigma] \in A$

Realizability algebras (2/2)

[Krivine'10]

Definition

A realizability algebra $\mathscr{A}$ is given by :

- 3 sets $\boldsymbol{\Lambda}$ ($\mathscr{A}$-terms), $\boldsymbol{\Pi}$ ($\mathscr{A}$-stacks), $\boldsymbol{\Lambda} \star \boldsymbol{\Pi}$ ($\mathscr{A}$-processes)
- 3 functions $(\cdot) : \boldsymbol{\Lambda} \times \boldsymbol{\Pi} \to \boldsymbol{\Pi}$, $(\star) : \boldsymbol{\Lambda} \times \boldsymbol{\Pi} \to \boldsymbol{\Lambda} \star \boldsymbol{\Pi}$, $(\mathsf{k}_{\_}) : \boldsymbol{\Pi} \to \boldsymbol{\Lambda}$
- A compilation function $(t, \sigma) \mapsto t[\sigma]$ into the set $\boldsymbol{\Lambda}$ of $\mathscr{A}$-terms
- A subset $\mathrm{PL} \subseteq \boldsymbol{\Lambda}$ (of proof-like $\mathscr{A}$-terms) such that for all (t, σ) :

$$\text{If } \operatorname{cod}(\sigma) \subseteq \mathrm{PL}, \quad \text{then} \quad t[\sigma] \in \mathrm{PL} \qquad (FV(t) \subseteq \operatorname{dom}(\sigma))$$

- A set of $\mathscr{A}$-processes $\perp\!\!\!\perp \subseteq \boldsymbol{\Lambda} \star \boldsymbol{\Pi}$ (the pole) such that :

$$\begin{array}{rcl}
\sigma(x) \star \pi \in \perp\!\!\!\perp & \text{implies} & x[\sigma] \star \pi \in \perp\!\!\!\perp \\
t[\sigma, x := a] \star \pi \in \perp\!\!\!\perp & \text{implies} & (\lambda x \,.\, t)[\sigma] \star a \cdot \pi \in \perp\!\!\!\perp \\
t[\sigma] \star u[\sigma] \cdot \pi \in \perp\!\!\!\perp & \text{implies} & (tu)[\sigma] \star \pi \in \perp\!\!\!\perp \\
a \star \mathsf{k}_\pi \cdot \pi \in \perp\!\!\!\perp & \text{implies} & \text{cc}[\sigma] \star a \cdot \pi \in \perp\!\!\!\perp \\
a \star \pi \in \perp\!\!\!\perp & \text{implies} & \mathsf{k}_\pi \star a \cdot \pi' \in \perp\!\!\!\perp
\end{array}$$

Canonical example : the λ_c-calculus

Terms, stacks and processes

Instructions	κ	$::=$	$\text{cc} \mid \text{quote} \mid \cdots$
Terms	t, u	$::=$	$x \mid \lambda x\,.\,t \mid tu \mid \kappa \mid \mathrm{k}_\pi$
Stacks	π, π'	$::=$	$\alpha \mid t \cdot \pi$ $\quad(\alpha \in \Pi_0,\ t$ closed$)$
Processes	p, q	$::=$	$t \star \pi$ $\quad(t$ closed$)$

- $\mathbf{\Lambda}, \mathbf{\Pi}, \mathbf{\Lambda} \star \mathbf{\Pi}$ = sets of closed terms, stacks, processes
- Compilation $t[\sigma]$ = substitution
- PL = set of closed terms containing no k_π
- $\perp\!\!\!\perp$ = any set of processes closed under anti-evaluation

Variant : the combinatory λ_c-calculus (1/2)

Terms, stacks and processes

Instructions	κ	$::=$	$\mathbf{I} \mid \mathbf{C} \mid \mathbf{B} \mid \mathbf{K} \mid \mathbf{W} \mid \text{œ} \mid \cdots$	
Terms	t, u	$::=$	$x \mid \kappa \mid tu \mid \mathrm{k}_\pi$	
Stacks	π, π'	$::=$	$\alpha \mid t \cdot \pi$	($\alpha \in \Pi_0$, t closed)
Processes	p, q	$::=$	$t \star \pi$	(t closed)

Krivine Abstract Machine (KAM)

I	$\mathbf{I} \star t \cdot \pi$	$\succ$	$t \star \pi$
K	$\mathbf{K} \star t \cdot u \cdot \pi$	$\succ$	$t \star \pi$
W	$\mathbf{W} \star t \cdot u \cdot \pi$	$\succ$	$t \star u \cdot u \cdot \pi$
C	$\mathbf{C} \star t \cdot u \cdot v \cdot \pi$	$\succ$	$t \star v \cdot u \cdot \pi$
B	$\mathbf{B} \star t \cdot u \cdot v \cdot \pi$	$\succ$	$t \star (uv) \cdot \pi$
Push	$tu \star \pi$	$\succ$	$t \star u \cdot \pi$
Save	$\text{œ} \star u \cdot \pi$	$\succ$	$u \star \mathrm{k}_\pi \cdot \pi$
Restore	$\mathrm{k}_\pi \star u \cdot \pi'$	$\succ$	$u \star \pi$
	$\cdots$		$\cdots$

Variant : the combinatory λ_c-calculus (2/2)

- Abstraction $\lambda^* x\,.\,t$ is defined from binary abstraction $\langle \lambda^* x\,.\,t \mid r\rangle$:

Definition of $\langle \lambda^* x\,.\,t \mid r\rangle$

$$
\begin{array}{rcll}
\langle \lambda^* x\,.\,t \mid r\rangle & \equiv & \mathbf{K}\,(r\,t) & (x \notin FV(t)) \\
\langle \lambda^* x\,.\,x \mid r\rangle & \equiv & r & \\
\langle \lambda^* x\,.\,t_1 t_2 \mid r\rangle & \equiv & \langle \lambda^* x\,.\,t_2 \mid \mathbf{B}\,r\,t_1\rangle & (x \notin FV(t_1),\ x \in FV(t_2)) \\
\langle \lambda^* x\,.\,t_1 t_2 \mid r\rangle & \equiv & \langle \lambda^* x\,.\,t_1 \mid \mathbf{C}\,(\mathbf{B}\,r)\,t_2\rangle & (x \in FV(t_1),\ x \notin FV(t_2)) \\
\langle \lambda^* x\,.\,t_1 t_2 \mid r\rangle & \equiv & \mathbf{W}\,\langle \lambda^* x\,.\,t_2 \mid \mathbf{C}\,\langle \lambda^* x\,.\,t_1 \mid \mathbf{B}\,r\rangle\rangle & (x \in FV(t_1),\ x \in FV(t_2))
\end{array}
$$

Lemma

For all t, u, r, π : $\quad \langle \lambda^* x\,.\,t \mid r\rangle \star u\cdot\pi \ \succ\ r \star t\{x := u\}\cdot\pi$

- Then we let : $\quad \lambda^* x\,.\,t \ \equiv\ \langle \lambda^* x\,.\,t \mid \mathbf{I}\rangle$

Lemma

For all t, u, π : $\quad \lambda^* x\,.\,t \star u\cdot\pi \ \succ\ t\{x := u\} \star \pi$

- Compilation function defined as expected, compiling λ as λ^*

Turning Boolean algebras into realizability algebras

- From a Boolean algebra $\mathbb{B}$, we can build a realizability algebra $\mathscr{A} = (\mathbf{\Lambda}, \mathbf{\Pi}, \mathbf{\Lambda} \star \mathbf{\Pi}, \ldots, \perp\!\!\!\perp)$, letting :
  - $\mathbf{\Lambda} \;=\; \mathbf{\Pi} \;=\; \mathbf{\Lambda} \star \mathbf{\Pi} \;=\; \mathbb{B}$
  - $b_1 \cdot b_2 \;=\; b_1 \star b_2 \;=\; b_1 b_2, \quad \mathsf{k}_b \;=\; b$
  - $\mathsf{PL} = \{1\}$
  - $t[\sigma] \;=\; \prod_{x \in FV(t)} \sigma(x)$
  - $\perp\!\!\!\perp = \{0\}$
- In the case where $\mathbb{B}$ is complete, the realizability model $\mathscr{M}^{(\mathscr{A})}$ is elementarily equivalent to the Boolean-valued model $\mathscr{M}^{(\mathbb{B})}$

  If $\mathbb{B}$ is not complete, then $\mathscr{A}$ automatically completes $\mathbb{B}$

Plan

(blackboard)